KINEMATIC AND DYNAMIC MODELING, ANALYSIS AND CONTROL OF ROBOTIC SYSTEMS VIA GENERALIZED COORDINATE TRANSFORMATION

By

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TO MY PARENTS

Thanks for everything!

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This work presents a unified approach to the dynamic modeling and analysis of the general case of rigid-link multi-degree of freedom mechanical devices and includes the detailed treatment of the specific case of the serial manipulator. The approach is based on the transference of the system dependence from one set of generalized coordinates to another (e.g., from the relative joint angles to cartesian referenced hand coordinates of the serial manipulator) and is shown to allow the analysis of any single-loop mechanism (e.g., the Bricard mechanism), multi-loop parallel-input linkages, systems with a superabundance kinematically independent inputs (e.g., redundant manipulators), and systems containing a superabundance of kinematically dependent inputs (e.g., walking machines). The technique involves the initial modeling of the system (or its components) in terms of simple open kinematic chain relationships; then using the

concept of loop closure and the kinematic constraints relating the possible sets of generalized coordinates the final desired system model is obtained.

The derivation of the initial dynamic model is based almost entirely on the principle of virtual work and the generalized principle of d'Alembert and is treated in great detail (from first-order properties through the linearized state space model, including actuator dynamics). The resultant model is expressed in terms of kinematic and dynamic influence coefficients and is particularly well suited for the transfer of generalized coordinates, especially the quadratic form of the non-linear effective load terms. The validity of the results of the unified modeling approach is demonstrated by some simple yet sufficient examples.

CHAPTER I

INTRODUCTION

One of the most pressing issues facing engineers today is the control of highly integrated mechanical-based systems capable of addressing a wide variety of tasks, without physical alteration of the system itself. These integrated systems (commonly referred to as robotic devices) are, in essence, asked to manipulate an object (e.g., a tool attached to the end-effector) in the presence of varying process and environmental effects in a very precise manner. To respond to this desire for both diverse motion capability and precise maintenance (during operation) of the specified motion requires not simply a battery of sensors feeding back infinite information to a central-processing-unit which magically assimilates and transforms these data into appropriate compensatory commands to the system actuators, but an acute awareness of the device itself. That is to say, a sufficient mathematical representation (model) of the object of control (the actuated mechanical linkage) must exist. The obtainment, and subsequent investigation of this model (which not only removes the magic from the control process, but also indicates what feedback information is necessary and sufficient for that process) is the primary focus of this work. While there is argument that one only

needs a qualitative model (e.g., knowledge that a particle's acceleration is linearly related to force) of the system for feed-back compensation, there is no question that a quantitative model (e.g., a particle's acceleration is equal to the force applied to the particle divided by its mass) is essential for feed-forward compensation. Now, while the author is not familiar with all the vagarities of adaptive control, it seems only reasonable that one would prefer to accomplish as much feed-forward compensation as possible, and then use feed-back techniques to obtain the (reduced) remaining required compensation. Admittedly, the quantification of the model (e.g., determination of the particle's mass) is considerably more difficult than its qualification, but this difficulty is not sufficient reason to dismiss the possibility (and potential) of combined feed-forward, feed-back control schemes for robotic devices. Keeping the goal of this combined control in mind, the kinematic and dynamic models developed in this work are derived in a qualitative manner while the final expressions require the quantification of basic system properties, such as link dimensions and mass parameters.

The formalized modeling procedure presented establishes a base technology capable of addressing the full range of possible mechanisms, from a single generic approach. The procedure stems from the work of Tesar and his graduate students, most notably the general development of Benedict and Tesar (1971) and the rigid-link serial manipulator model

of Thomas (1981). The model is based on the use of kinematic influence coefficients. These coefficients describe the position-dependent reaction of such pertinent systems parameters as link centroids to the action of the independent generalized coordinates (system inputs). While the basic model is a rigid-link model, and robotic devices typically demonstrate some compliance (Sunada and Dubowsky, 1983), the model is presently being used as the basis for extension to the treatment of quasi-static deformations (Fresonke, 1985) and quasi-statically induced vibrations (Behi, 1985), for optimized design procedures (Thomas, 1984 and 1985), and as an aid in the experimental identification of pertinent system parameters.

The extension to nonrigid-link devices is not specifically addressed here, nor is the question of real-time dynamic compensation (see Wander, 1985). Again, the primary objective of this work is the development of a general, unified modeling procedure capable of addressing the full range of robotic devices, including redundant manipulators, cooperating robots, walking machines and multi-fingered end-effectors. For discussion and comparison of the various popular dynamic model formulations, the reader is referred to Tesar and Thomas (1979) and Thomas and Thomas (1982b), Silver (1982) and Lee (1982), among others.

Before giving a more detailed overview of the contents of this work, it will prove beneficial to introduce the notational scheme developed herein. Referring (throughout)

to Table 1-1, the basic setup involves a block arrangement consisting of a central block surrounded by both pre- and post-, superscripts and subscripts. The center block is reserved for a symbol representing either a system parameter(s) (e.g., the set ($\underline{u}$)), a physical quantity (e.g., a vector of applied loads ($\underline{T}$) or a mathematical operator or operation (e.g., partial differentiation). Next, the two superscript blocks are reserved for dependent system parameters (or their properties, such as velocity), with the post-script indicating which parameter(s) is involved and with the pre-script giving additional information concerning the parameter(s). Finally, the two subscript blocks are used in exactly the same fashion as the superscripts, but are reserved for independent system parameters (e.g., generalized coordinates ($\underline{\varphi}$) or fixed linkage dimensions (a_{jk})). At this stage the reader is again referred to Table 1.1 to review the illustrative examples. While this notation is indeed redundant when dealing with a single (fixed) set of independent generalized coordinates, it is far from redundant when dealing with the analytic developments and applications in Chapters III and IV where the system dependence is transferred from one possible generalized coordinate set to another. Regardless, it is felt that even when dealing with a fixed input set, that the separation of dependent and independent system parameters by the respective use of superscripts and subscripts serves as a valuable aid in the description and analysis of the system

Table 1-1. Notation

Dependent	[Modifier]		[Parameter(s)]	Dependent
↕	– – – –	Symbol Operator	– – – –	↕
Independent	[Modifier]		[Parameter(s)]	Independent

Examples

1. $\underline{u} = f(\varphi)$:

$$\underline{u} = (u^1, u^2, \ldots, u^P)^T \text{ - dependent } \quad \begin{matrix} [\ ] & & [p] \\ & [u] & \\ [\ ] & & [\ ] \end{matrix}$$

$$\underline{\varphi} = (\varphi_1, \varphi_2, \ldots \varphi_M)^T \text{ - independent } \quad \begin{matrix} [\ ] & & [\ ] \\ & [\varphi] & \\ [\ ] & & [m] \end{matrix}$$

2. $$\frac{\partial(\underline{u})}{\partial\underline{\varphi}} = \begin{matrix} [\ ] & & [\ \underline{u}] \\ & \frac{(\ )}{(\ )} & \\ [\ ] & & [\underline{\varphi}] \end{matrix} \equiv [G^u_\varphi] = [G_u] \text{ (Thomas amd Tesar, 1982b)}$$

where $G \equiv \frac{\partial(\ )}{\partial(\ )}$ [See eqn's. (2-5) and (2-6)]

3. $$\frac{\partial^2(\underline{u})}{\partial\underline{\varphi}\partial\underline{\varphi}} = \begin{matrix} [\ ] & & [\underline{u}] \\ & \frac{\partial^2(\ )}{\partial\underline{\varphi}\partial\underline{\varphi}} & \\ [\ ] & & [\underline{\varphi\varphi}] \end{matrix} \equiv [H^u_{\varphi\varphi}] = [H_u] \text{ (Thomas and Tesar, 1982b)}$$

where $H \equiv \frac{\partial(\ )}{\partial(\ )\partial(\ )}$ [See eqn's. (2-11)-(2-27)]

4. $$\sum_{j=1}^{M}\{M^{jk}[{}^jG^c_\varphi]^T[{}^jG^c_\varphi]\} = \begin{matrix} [P] & & [\ ] \\ & [I^*] & \\ [\ ] & & [\underline{\varphi\varphi}] \end{matrix} \equiv [{}^PI^*_{\varphi\varphi}]$$

where $${}^PI^* \equiv \sum_{j=1}^{M} M^{jk}[{}^jG^c_{(\ )}]^T[{}^jF^c_{(\ )}]$$ [See eqn. (2-157)]

under investigation. In addition to being descriptive in nature, this notational scheme underscores the fundamental theme of this work; that there are only two basic types of system parameters (dependent and independent), but that there is not a unique set of generalized coordinates from which to view the system. The notational scheme also has been developed with the assimilation of ten plus years of various kinematic and dynamic analyses presented by Tesar and his students into a single formalized unit. As a final note to readers familar with the notation utilized by Thomas and Tesar (1982b), to obtain the parameter descriptions given in that work simply drop all subscripts (since only one set of generalized coordinates is addressed) and rotate the superscripts one block in the clock-wise direction (e.g., the post-superscript in this work becomes the subscript in Thomas and Tesar (1982b) as shown in examples 2 and 3 of Table 1-1).

In keeping with the desire to present as generic a treatment as possible, Chapter II starts off with the development of the first-, second- and third-order kinematics of a general multi-degree of freedom system. Here, the conceptual use of kinematic influence coefficients, along with the definition of the associated notation, is introduced. The concept of influence coefficients is fundamental to the modeling methodology employed throughout this work and the need to understand this basic idea cannot be overemphasized. Additionally, the

qualitative approach to deriving the high-order kinematics is stressed, as well as the utility and simplicity of the standard Jacobian format for arranging the results of vector differentiation. Again, the reader's complete familiarity with the method of derivation employed (and the notation involved in describing the results) in this section (II.A.1.) will greatly facilitate his understanding of the remainder of this work.

Next, the kinematics of the general serial manipulator are addressed. First, the geometric parameters and the associated notation describing the serial manipulator are defined, then the first-, second- and third-order kinematics are addressed. The derivation presented here is based on the previously developed generalized kinematics, with the major emphasis here being on the determination of the kinematic influence coefficients for this specific device. The method of derivation and form of the results (simple vectors or vector cross products) are based directly on the work of Thomas (1981).

Having completed the treatment of the system kinematics, attention is then turned to the development of the dynamic model. The dynamic model presented here is almost completely determined with the use of just two principles of mechanics: the principle of virtual work and the generalized principle of D'Alembert. In light of this, and in keeping with the generic nature of this work, the use of these two principles in the development of the dynamic equations for a general

device is presented first. Here the momentum form of Newton's second law is used. After establishing the general approach, the dynamics of the serial manipulator are addressed specifically. In this treatment (II.B.4.), the results for the manipulator are expressed in matrix form, this being consistent with the method of derivation as well as with the computer language (APL) that the author uses for simulation. In fact, the author's familiarity with the multi-dimensional array handling and operational capabilities of this language (APL) is most certainly a contributing factor in the chosen form of the resultant modeling expressions. Again, for the investigator familiar with the work of Thomas (1981) (aside from the notation and method of derivation) and the expression of the coefficients of the dynamic model in matrix (vector) form here, as opposed to the summation (scalar) form of Thomas (1981), is the only difference in the modeling results of the two works. Also, and of paramount importance to the transformation of generalized coordinates technique presented in Chapter III, a generalized vector scalar product operator ($\bullet$) is introduced. Because of the significance of this operator to this entire work, it is addressed in detail in Appendix A.

Next, for completeness, the linearization of the dynamic equations of the general serial manipulator is pursued. The derivation of these linearized equations is again attacked in a qualitative fashion (i.e., partials are taken with

respect to the generalized coordinate positions, velocities and accelerations). Unfortunately, the resultant expressions are extremely complicated and a nice compact form was not found. At any rate, having obtained expressions for the linearized equations, the drive train kinematics and actuator (d-c motor) dynamics are incorporated into the model. The nominal voltages required to drive the system along some specified generalized coordinate trajectory are then determined and, finally, a velocity-referenced state space model of the linearized system is developed. While the resulting state space model can be used to address the possibility of applying modern linear control techniques to controller design for specific manipulator trajectories (Whitehead, 1984), this is not a goal of this work and is not pursued. This completes the material presented in Chapter II, covering the complete development of the kinematic and dynamic equations governing the motion of both a general mechanism and the serial manipulator in terms of a specific set of generalized coordinates.

Having established the general procedure for obtaining the dynamic model directly in terms of a specific set of generalized coordinates, attention is now turned to the determination of the model with respect to any set of generalized coordinates. Chapter III specifically addresses this question, with the basic approach employing a transfer of system dependence from one set of generalized coordinates to another. The technique involves the determination of the

model (in the direct fashion of Chapter II) with respect to some initial set of generalized coordinates (selected for ease of modeling), then (using this initial model information along with the principle of virtual work and the system's kinematic constraints) the system model is effectively transferred to any arbitrary desired set of generalized coordinates. The treatment here is basic in nature and covers the three main elements comprising the system modeling of Chapter II: kinematics, dynamics and the linearized state space representation. The use of this concept is not without precedent as it has been employed in one form or another by numerous researchers (referenced in Chapter III), primarily with regards to system control. One of the self-imposed constraints on the model transformation presented here is that the resultant desired model maintain the same general form as the initial model (e.g., equations (2-7), (2-17) and (2-198)). Also, as is shown in Chapter III, the generalized scalar product (Appendix A) is the key mathematical realization allowing the maintenance of the general form.

Now, having established the basic procedures for obtaining an initial system model (Chapter II) and the subsequent determination of the system model referenced to any set of generalized coordinates (Chapter III), the full utility of the generalized coordinate transfer technique is addressed. Through the treatment of some fairly general applications, Chapter IV presents a unified

and straight-forward approach to the kinematic and dynamic modeling of a wide variety of extremely complicated mechanical systems. This unified approach first entails the initial modeling of the system (or separable components of the system) in the relatively simple terms of an open-loop kinematic chain. Then, having obtained this simple initial model, the system is constrained (or closed) by single or multiple application of the transfer equations resulting in the desired dynamic model. This means that the most complicated kinematic device that one need model directly is the simplest possible, the open kinematic chain (or serial manipulator). Finally, for illustration of the full power and scope of this technique it is shown to be capable of modeling

1. Any single-loop mechanism (including such degenerate devices as the Bricard mechanism) with respect to any set of generalized coordinates.
2. Multi-loop parallel-input mechanisms, such as the generalized Stewart platform.
3. Systems with a superabundance of kinematically independent inputs, such as the redundant manipulator.
4. Systems with a superabundance of kinematically dependent inputs; such as cooperating robots, walking machines and multi-fingered end-effectors.

CHAPTER II

DEVELOPMENT OF THE CONTROLLING EQUATIONS

The investigation of the basic kinematic and dynamic nature of multi-degree of freedom mechanisms and the subsequent determination of a set of differential equations useful in addressing the control of such mechanisms are presented in this chapter. The presentation is separated into three major divisions. The first section develops a generalized approach to the modeling and analysis of the system kinematics. The approach, which is based on the use of kinematic influence coefficients, is applicable to mechanisms consisting of rigid, or at least quasi-rigid, links. In the second section, the influence coefficient methodology is incorporated in the determination of the dynamic model, yielding the relationship between the system's load and motion states. Finally, the dynamic equations (developed in the second section) are linearized and a general state space representation is presented. This representation, which includes actuator dynamics, is useful when considering the control of the mechanism about some prespecified nominal trajectory.

Method of Kinematic Influence Coefficients

This approach to dynamic modeling of rigid-link mechanisms is based on the separation of all kinematic (and dynamic) phenomena into a collection of purely position dependent functions (kinematic influence coefficients) operated on by independent functions of time (input time states). The concept of kinematic influence coefficients can be applied to all classes of mechanisms (e.g., parallel-input planar systems, serial-input spatial devices, etc.) and has been well established in the literature by the works of Tesar and his graduate students: Benedict and Tesar (1971), (1978a) and (1978b), Freeman and Tesar (1982) and (1984), Cox and Tesar (1981), Thomas (1981) and Sklar and Tesar (1984). Because of the fundamental nature of this concept and its almost limitless range of application, this section is presented in two parts. The first contains a detailed development, in general terms, of the basic methodology involved. The second addresses the specific case of the serial manipulator.

Generalized Kinematics

Consider a P-dimensional time varying vector

$$\underline{u}(t) = (u^1(t), \ldots, u^P(t))^T \tag{2-1}$$

representative of the motion parameters required to describe the kinematics of a given system where the superscripts indicate which parameter is involved (not that u^p is u raised to the p^{th} power). Further, consider that these parameters are functions of an M-dimensional set of generalized coordinates

$$\underline{\varphi}(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_M(t))^T \tag{2-2}$$

This allows a parametric description of the system where

$$u^p = u^p(\underline{\varphi}); p = 1,2,\ldots,P \tag{2-3}$$

and

$$\varphi_n = \varphi_n(t); n = 1,2,\ldots,M \tag{2-4}$$

Adopting the standard convention for differentiation of a vector (e.g.,$\underline{u}(\underline{\varphi})$) by a vector (e.g., $\underline{\varphi}(t)$) one obtains the typical Jacobian form, where the n^{th} column of the result is the partial derivative of $\underline{u}(\underline{\varphi})$ with respect to the n^{th} component of $\underline{\varphi}(t)$, for the first time derivative of $\underline{u}(t)$) as

$$\begin{aligned} \dot{\underline{u}} &= [\frac{\partial \underline{u}}{\partial \underline{\varphi}}]\dot{\underline{\varphi}} \\ &= [\frac{\partial \underline{u}}{\partial \varphi_1} \; \frac{\partial \underline{u}}{\partial \varphi_2} \; \ldots \; \frac{\partial \underline{u}}{\partial \varphi_M}]\dot{\underline{\varphi}} \\ &= \begin{bmatrix} \frac{\partial u^1}{\partial \varphi_1} & \frac{\partial u^1}{\partial \varphi_2} & \ldots & \frac{\partial u^1}{\partial \varphi_M} \\ & & & \\ \frac{\partial u^P}{\partial \varphi_1} & & & \frac{\partial u^P}{\partial \varphi_M} \end{bmatrix} \begin{matrix} \dot{\varphi}_1 \\ \varphi_2 \\ \vdots \\ \dot{\varphi}_M \end{matrix} \end{aligned} \tag{2-5}$$

Throughout this section an explicit shorthand notation is developed to serve as a descriptive aid in the expression of the system kinematics. The first order geometric

derivatives (i.e., kinematic influence coefficients) are denoted as follows:

$$[\frac{\partial \underline{u}}{\partial \underline{\varphi}}] \equiv [G^{u}_{\varphi}] = P \times M \text{ matrix}$$

$$[\frac{\partial \underline{u}}{\partial \varphi_n}] = [G^{u}_{\varphi\,;n}] \equiv [G^{u}_{n}] \equiv \underline{g}^{u}_{n} = P \times 1 \text{ vector}$$

$$[\frac{\partial u^p}{\partial \underline{\varphi}}] = [G^{u}_{\varphi_p;}] \equiv [G^{u}_{\varphi}] \equiv \underline{g}^{p}_{\varphi} = 1 \times M \text{ vector} \tag{2-6}$$

$$[\frac{\partial u^p}{\partial \varphi_n}] = [G^{u}_{\varphi_p;n}] \equiv [G^{p}_{n}] \equiv g^{p}_{n} = 1 \times 1$$

where the letter (g,G) is reserved for first order geometric derivatives, the superscript indicates the dependent parameter(s) involved (i.e., u^p) and the subscript specifies which generalized coordinate(s) (i.e., φ_n) is involved. Also, the shape of the result is indicated by the dimensions of symbols used as the superscript and subscript(s). For example, the shape of $\underline{g}^{u}_{n}$ is equal to the dimension of the subscript u (i.e., P) by the dimension of the subscript n (i.e., 1). Referring to equations (2-6), the first order kinematics of the system can be written in a variety of ways depending on the manner of expression one wishes to employ. The velocity ($\dot{\underline{u}}$) of the complete parameter set ($\underline{u}$) can be expressed as

$$\dot{\underline{u}} = [G^{u}_{\varphi}]\dot{\underline{\varphi}} = (P \times M)(M \times 1) = P \times 1 \tag{2-7}$$

or

$$\dot{\underline{u}} = \sum_{n=1}^{M} \underline{g}^{u}_{n}\,\dot{\varphi}_n = \Sigma(P \times 1)(1 \times 1) = P \times 1 \text{ vector} \tag{2-8}$$

And, the velocity ($\dot{u}^p$) of a particular dependent parameter (u^p) is written as

$$\dot{u}^p = \underline{g}_\varphi^p \, \dot{\underline{\varphi}} = (1 \times M)(M \times 1) = 1 \times 1 \qquad (2\text{-}9)$$

or

$$\dot{u}^p = \sum_{n=1}^{M} g_n^p \, \dot{\varphi}_n = \Sigma(1 \times 1)(1 \times 1) = 1 \times 1 \qquad (2\text{-}10)$$

Noting that first-order influence coefficients (g,G) are functions of the generalized coordinates ($\underline{\varphi}$), one recognizes that the velocity vector ($\dot{\underline{u}}$) is a function both of the position ($\underline{\varphi}$) and velocity ($\dot{\underline{\varphi}}$) of the independent coordinates (φ_m; m = 1,2,...,M). This allows one to obtain the second time derivative ($\ddot{\underline{u}}$) of the dependent parameters ($\underline{u}$) from

$$\ddot{\underline{u}} = \frac{\partial \dot{\underline{u}}}{\partial \dot{\underline{\varphi}}} \ddot{\underline{\varphi}} + \frac{\partial \dot{\underline{u}}}{\partial \underline{\varphi}} \dot{\underline{\varphi}} = \ddot{\underline{u}}_{\ddot{\varphi}} = \ddot{\underline{u}}_{\dot{\varphi}\dot{\varphi}} \qquad (2\text{-}11)$$

The first partial derivative in equation (2-11) is

$$\frac{\partial \dot{\underline{u}}}{\partial \dot{\underline{\varphi}}} = [G_{\dot{\varphi}}^{\dot{u}}] = \frac{\partial}{\partial \dot{\underline{\varphi}}}([G_\varphi^u]\dot{\underline{\varphi}}) = [G_\varphi^u]\frac{\partial \dot{\underline{\varphi}}}{\partial \dot{\underline{\varphi}}} = [G_\varphi^u][I]_{MxM} \qquad (2\text{-}12)$$

where the last term is the MxM identity matrix. This gives the partial acceleration of ($\underline{u}$) due to the acceleration of the inputs ($\underline{\varphi}$) as

$$\ddot{\underline{u}}_{\ddot{\varphi}} = [G_\varphi^u]\ddot{\underline{\varphi}} \qquad (2\text{-}13)$$

Recalling the standard Jacobian form for derivatives of vectors, one has

$$\frac{\partial \underline{\dot{u}}}{\partial \underline{\underline{\varphi}}} = [G_{\varphi}^{\dot{u}}] = \frac{\partial([G_{\varphi}^{u}]\underline{\dot{\varphi}})}{\partial \underline{\varphi}} = \begin{bmatrix} \frac{\partial \dot{u}^1}{\partial \varphi_1} & \frac{\partial \dot{u}^1}{\partial \varphi_2} & \cdots & \frac{\partial \dot{u}^1}{\partial \varphi_M} \\ \frac{\partial \dot{u}^2}{\partial \varphi_1} & & & \\ \vdots & & & \\ \frac{\partial \dot{u}^P}{\partial \varphi_1} & & & \frac{\partial \dot{u}^P}{\partial \varphi_M} \end{bmatrix} \tag{2-14}$$

where

$$g_m^{\dot{p}} = \frac{\partial \dot{u}^p}{\partial \varphi_m} = \frac{\partial}{\partial \varphi_m} \left(\sum_{n=1}^{M} g_n^p \, \dot{\varphi}_n \right), \quad p = 1,2,\ldots,P \quad m = 1,2,\ldots,M \tag{2-15}$$

Recalling the definition given in equation (2-6) for (g_n^p), and performing the differentiation, equation (2-15) becomes

$$\begin{aligned} g_m^{\dot{p}} &= \sum_{n=1}^{M} \left[\frac{\partial}{\partial \varphi_m} \left(\frac{\partial u^p}{\partial \varphi_n} \right) \right] \dot{\varphi}_n \\ &= \sum_{n=1}^{M} h_{mn}^p \, \dot{\varphi}_n \\ &= 1 \times 1 \end{aligned} \tag{2-16}$$

where

$$\frac{\partial}{\partial \varphi_m} \left(\frac{\partial u^p}{\partial \varphi_n} \right) \equiv h_{mn}^p = 1 \times 1 \times 1 \tag{2-17}$$

Alternately, equation (2-16) can be expressed in vector form as

$$g_m^{\dot{p}} = \underline{\dot{\varphi}}^T (\underline{h}_{m\varphi}^p)^T = 1 \times 1 \tag{2-18}$$

where

$$\underline{h}_{m\varphi}^p = (h_{m1}^p, h_{m2}^p, \ldots, h_{mM}^p) = 1 \times 1 \times M \tag{2-19}$$

Here, the letter (h,H) is used to denote the second-order geometric derivative, and the shape of the result is equal to the dimension of the superscript by the dimensions of the subscripts ordered from left to right. Using the vector form of equation (2-18), the p^{th} row of equation (2-14) can be written as

$$\begin{aligned}
\left[\frac{\partial \dot{\underline{u}}}{\partial \underline{\varphi}}\right]_{p;} &= [G^{\dot{u}}_{\varphi}]_{p;} = [G^{\dot{p}}_{\varphi}] \\
&= \dot{\underline{\varphi}}^T[(\underline{h}^p_{1\varphi})\ (\underline{h}^p_{2\varphi})\ \ldots\ (\underline{h}^p_{M\varphi})^T] \\
&= \dot{\underline{\varphi}}^T[H^p_{\varphi\varphi}]^T \qquad (2\text{-}20) \\
&= 1 \times M
\end{aligned}$$

where

$$[H^p_{\varphi\varphi}] \equiv \left[\frac{\partial}{\partial \underline{\varphi}}\left(\frac{\partial u^p}{\partial \underline{\varphi}}\right)\right] = \begin{bmatrix} h^p_{11} & h^p_{12} & \ldots & h^p_{1M} \\ h^p_{21} & & & \\ \vdots & & & \\ h^p_{M1} & & & h^p_{MM} \end{bmatrix} = 1 \times M \times M \qquad (2\text{-}21)$$

From this, the partial acceleration of the p^{th} dependent parameter (u^p) due to the velocity of the inputs ($\underline{\varphi}$) is seen to be

$$\ddot{u}^p_{\dot{\varphi}\dot{\varphi}} = \dot{\underline{\varphi}}^T[H^p_{\varphi\varphi}]^T\dot{\underline{\varphi}} = 1 \times 1 \times 1 \qquad (2\text{-}22)$$

Since the transpose of a scalar is equal to itself, equation (2-22) becomes

$$\ddot{u}^{p}_{\dot{\varphi}\dot{\varphi}} = \underline{\dot{\varphi}}^{T}[H^{p}_{\varphi\varphi}]\underline{\dot{\varphi}} \tag{2-23}$$

Now, considering the complete parameter set ($\underline{u}$), one has

$$\underline{\ddot{u}}_{\dot{\varphi}\dot{\varphi}} = \begin{pmatrix} \underline{\dot{\varphi}}^{T}[H^{1}_{\varphi\varphi}]\underline{\dot{\varphi}} \\ \underline{\dot{\varphi}}^{T}[H^{2}_{\varphi\varphi}]\underline{\dot{\varphi}} \\ \vdots \\ \underline{\dot{\varphi}}^{T}[H^{P}_{\varphi\varphi}]\underline{\dot{\varphi}} \end{pmatrix} = \underline{\dot{\varphi}}^{T}[H^{u}_{\varphi\varphi}]\underline{\dot{\varphi}} = P \times 1 \tag{2-24}$$

where

$$[H^{u}_{\varphi\varphi}] = P \times M \times M \tag{2-25}$$

and

$$[H^{u}_{\varphi\varphi}]_{p;;} = [H^{p}_{\varphi\varphi}] \tag{2-26}$$

Finally, from equations (2-11), (2-13) and (2-24), the acceleration ($\underline{u}$) is found to be

$$\underline{\ddot{u}} = [G^{u}_{\varphi}]\underline{\ddot{\varphi}} + \underline{\dot{\varphi}}^{T}[H^{u}_{\varphi\varphi}]\underline{\dot{\varphi}} \tag{2-27}$$

Observing that the acceleration ($\underline{\ddot{u}}$) is a function of ($\underline{\varphi}$), ($\underline{\dot{\varphi}}$) and ($\underline{\ddot{\varphi}}$), the third order time derivative ($\underline{\dddot{u}}$) can be determined from

$$\underline{\dddot{u}} = \frac{\partial \underline{\ddot{u}}}{\partial \underline{\ddot{\varphi}}}\,\underline{\dddot{\varphi}} + \frac{\partial \underline{\ddot{u}}}{\partial \underline{\dot{\varphi}}}\,\underline{\ddot{\varphi}} + \frac{\partial \underline{\ddot{u}}}{\partial \underline{\varphi}}\,\underline{\dot{\varphi}} \tag{2-28}$$

Here

$$\frac{\partial \underline{\ddot{u}}}{\partial \underline{\ddot{\varphi}}} = [G^u_{\varphi}]\frac{\partial \underline{\ddot{\varphi}}}{\partial \underline{\ddot{\varphi}}} = [G^u_{\varphi}][I]_{M\times M} \tag{2-29}$$

so

$$\frac{\partial \underline{\ddot{u}}}{\partial \underline{\ddot{\varphi}}}\,\underline{\ddot{\varphi}} = [G^u_{\varphi}]\underline{\ddot{\varphi}} \tag{2-30}$$

The second term in equation (2-28) is found from

$$\frac{\partial \underline{\ddot{u}}}{\partial \underline{\dot{\varphi}}} = \frac{\partial \underline{\ddot{u}}_{\dot{\varphi}\dot{\varphi}}}{\partial \underline{\dot{\varphi}}} = \begin{bmatrix} \frac{\partial \ddot{u}^1_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}_1} & \frac{\partial \ddot{u}^1_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}_2} & \cdots & \frac{\partial \ddot{u}^1_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}_M} \\ \frac{\partial \ddot{u}^2_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}_1} & & & \\ \vdots & & & \\ \frac{\partial \ddot{u}^P_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}_1} & & & \frac{\partial \ddot{u}^P_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}^M} \end{bmatrix} \tag{2-31}$$

where, from equation (2-22),

$$\begin{aligned} \frac{\partial \ddot{u}^p_{\dot{\varphi}\dot{\varphi}}}{\partial \dot{\varphi}_1} &= \frac{\partial(\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}]^T\underline{\dot{\varphi}})}{\partial \dot{\varphi}_1} \\ &= ([H^p_{\varphi\varphi}]^T\underline{\dot{\varphi}})_{1;} + (\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}]^T)_{;1} \\ &= (\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}])_{;1} + (\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}]^T)_{;1} \end{aligned} \tag{2-32}$$

since

$$(\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}])_{;1} = [([H^p_{\varphi\varphi}]^T\underline{\dot{\varphi}})_{1;}]^T = 1 \times 1 \tag{2-33}$$

The pth row of equation (2-31) can be written as

$$\begin{aligned} \frac{\partial \ddot{u}^p_{\dot{\varphi}\dot{\varphi}}}{\partial \underline{\dot{\varphi}}} &= ((\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}])_{;1} + (\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}]^T)_{;1}, \ldots, (\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}])_{;M} + \\ &\qquad (\underline{\dot{\varphi}}^T[H^p_{\varphi\varphi}]^T)_{;M}) \\ &= \underline{\dot{\varphi}}^T[[H^p_{\varphi\varphi}] + [H^p_{\varphi\varphi}]^T] \end{aligned} \tag{2-34}$$

The complete Jacobian of equation (2-31) is then

$$\frac{\partial \underline{\ddot{u}}}{\partial \underline{\dot{\Phi}}} = \begin{bmatrix} \underline{\dot{\varphi}}^T[[H^1_{\varphi\varphi}] + [H^1_{\varphi\varphi}]^T] \\ \hline \underline{\dot{\varphi}}^T[[H^2_{\varphi\varphi}] + [H^2_{\varphi\varphi}]^T] \\ \hline \vdots \\ \hline \underline{\dot{\varphi}}^T[[H^P_{\varphi\varphi}] + [H^P_{\varphi\varphi}]^T] \end{bmatrix} \tag{2-35}$$

$$\equiv \underline{\dot{\varphi}}^T[[H^u_{\varphi\varphi}] + [H^u_{\varphi\varphi}]^T]$$

yielding

$$\frac{\partial \underline{\ddot{u}}}{\partial \underline{\dot{\Phi}}} \underline{\ddot{\varphi}} = \underline{\dot{\varphi}}^T[[H^u_{\varphi\varphi}] + [H^u_{\varphi\varphi}]^T]\underline{\ddot{\varphi}} \tag{2-36}$$

where the transpose operation is performed plane by plane. The last partial in equation (2-28) can be separated into two parts, giving

$$\frac{\partial \underline{\ddot{u}}}{\partial \underline{\varphi}} = \frac{\partial \underline{\ddot{u}}_{\ddot{\varphi}}}{\partial \underline{\varphi}} + \frac{\partial \underline{\ddot{u}}_{\dot{\varphi}\dot{\varphi}}}{\partial \underline{\varphi}} \tag{2-37}$$

where, recalling equations (2-13) through (2-20),

$$\frac{\partial \underline{\ddot{u}}_{\ddot{\varphi}}}{\partial \underline{\varphi}} = \frac{\partial([G^u_{\varphi}]\underline{\ddot{\varphi}})}{\partial \underline{\varphi}}$$

$$= \begin{bmatrix} \underline{\ddot{\varphi}}^T[H^1_{\varphi\varphi}]^T \\ \hline \underline{\ddot{\varphi}}^T[H^2_{\varphi\varphi}]^T \\ \hline \vdots \\ \hline \underline{\ddot{\varphi}}^T[H^P_{\varphi\varphi}]^T \end{bmatrix} \tag{2-38}$$

$$\equiv \underline{\ddot{\varphi}}^T [H^u_{\varphi\varphi}]^T = P \times M$$

and

$$\frac{\partial \underline{\ddot{u}}_{\dot{\varphi}\dot{\varphi}}}{\partial \underline{\varphi}} = \frac{\partial(\underline{\dot{\varphi}}^T [H^u_{\varphi\varphi}]^T \underline{\dot{\varphi}})}{\partial \underline{\varphi}}$$

$$= \begin{bmatrix} \frac{\partial(\underline{\dot{\varphi}}^T [H^1_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_1} & \frac{\partial(\underline{\dot{\varphi}}^T [H^1_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_2} & \cdots & \frac{\partial(\underline{\dot{\varphi}}^T [H^1_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_M} \\ \frac{\partial(\underline{\dot{\varphi}}^T [H^2_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_1} & & & \\ \vdots & & & \\ \frac{\partial(\underline{\dot{\varphi}}^T [H^P_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_1} & & & \frac{\partial(\underline{\dot{\varphi}}^T [H^P_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_M} \end{bmatrix} \quad (2\text{-}39)$$

Now

$$\frac{\partial(\underline{\dot{\varphi}}^T [H^p_{\varphi\varphi}] \underline{\dot{\varphi}})}{\partial \varphi_1} = \underline{\dot{\varphi}}^T [\frac{\partial}{\partial \varphi_1} ([H^p_{\varphi\varphi}])] \underline{\dot{\varphi}}$$

$$\equiv \underline{\dot{\varphi}}^T [D^p_{1\varphi\varphi}] \underline{\dot{\varphi}} = 1 \times 1 \quad (2\text{-}40)$$

yielding

$$\frac{\partial u_{\underline{\varphi\varphi}}}{\partial \underline{\varphi}} = \begin{bmatrix} \underline{\dot{\varphi}}^T [D^1_{1\varphi\varphi}] \underline{\dot{\varphi}} & \underline{\dot{\varphi}}^T [D^1_{2\varphi\varphi}] \underline{\dot{\varphi}} & \cdots & \underline{\dot{\varphi}}^T [D^1_{M\varphi\varphi}] \underline{\dot{\varphi}} \\ \underline{\dot{\varphi}}^T [D^2_{1\varphi\varphi}] \underline{\dot{\varphi}} & & & \\ \vdots & & & \\ \underline{\dot{\varphi}}^T [D^P_{1\varphi\varphi}] \underline{\dot{\varphi}} & & & \underline{\dot{\varphi}}^T [D^P_{M\varphi\varphi}] \underline{\dot{\varphi}} \end{bmatrix} \quad (2\text{-}41)$$

$$\equiv \underline{\varphi}^T [D^u_{\varphi\varphi\varphi}] \underline{\varphi} = P \times M$$

The letter (d,D) is used to denote the third-order geometric derivative, and the shape of the quantity is equal to the dimension of the superscript by the dimensions of the subscripts, respectively. Further defining the third order kinematic influence coefficient (d,D) one has

$$\frac{\partial^3 \underline{u}}{\partial \underline{\varphi}^3} = [D^u_{\varphi\varphi\varphi}] = P \times M \times M \times M$$

$$\frac{\partial^3 u^p}{\partial \underline{\varphi}^3} = [D^u_{\varphi\varphi\varphi}]_{p;;;} = [D^p_{\varphi\varphi\varphi}] = 1 \times M \times M \times M \tag{2-42}$$

$$\frac{\partial}{\varphi_l}\left(\frac{\partial^2 \underline{u}}{\partial \underline{\varphi}^2}\right) = [D^u_{\varphi\varphi\varphi}]_{;l;;} = [D^u_{l\varphi\varphi}] = P \times 1 \times M \times M$$

$$\frac{\partial}{\partial \varphi_l}\left(\frac{\partial}{\partial \varphi_m}\left(\frac{\partial u^p}{\partial \varphi_n}\right)\right) = [D^u_{\varphi\varphi\varphi}]_{p;l;m;n;} = d^p_{lmn} = 1 \times 1 \times 1 \times 1$$

Now, substituting equations (2-38) and (2-41) into equation (2-37) gives the partial derivative of the dependent parameter acceleration ($\ddot{\underline{u}}$) with respect to the generalized coordinate(s) position ($\underline{\varphi}$) as

$$\frac{\partial \ddot{\underline{u}}}{\partial \underline{\varphi}} = \ddot{\underline{\varphi}}^T [H^u_{\varphi\varphi}]^T + \dot{\underline{\varphi}}^T [D^u_{\varphi\varphi\varphi}] \dot{\underline{\varphi}} \tag{2-43}$$

Finally, from equations (2-28), (2-30), (2-36) and (2-43), the third-order time derivative ($\dddot{\underline{u}}$) of the dependent parameter set ($\underline{u}$) is

$$\dddot{\underline{u}} = [G^u_\varphi]\dddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T([H^u_{\varphi\varphi}] + [H^u_{\varphi\varphi}]^T)\ddot{\underline{\varphi}} + (\ddot{\underline{\varphi}}^T[H^u_{\varphi\varphi}]^T + \dot{\underline{\varphi}}^T[D^u_{\varphi\varphi\varphi}]\dot{\underline{\varphi}})\dot{\underline{\varphi}}$$

$$= [G^u_\varphi]\dddot{\underline{\varphi}} + \ddot{\underline{\varphi}}^T(H^u_{\varphi\varphi}] + 2[H^u_{\varphi\varphi}]^T)\dot{\underline{\varphi}} + (\dot{\underline{\varphi}}^T[D^u_{\varphi\varphi\varphi}]\dot{\underline{\varphi}})\dot{\underline{\varphi}} \tag{2-44}$$

completing the general kinematic development.

Kinematics of Serial Manipulators

The analytical development presented here, altered notationally and extended to cover third-order kinematics, is based directly on the work of Thomas (1981) and is included

for completeness. The definition of the particular physical system that is being considered is addressed first. Then the notation defining the physical parameters is established. Next, the geometric relationships corresponding to equation (2-3) are developed. Finally, the kinematics of the serial manipulator are derived in terms of explicit equations defining the kinematic influence coefficients as opposed to the conceptual relationships of the preceding general development.

System definition and notation

The analytical development presented here, altered notationally and extended to cover third-order kinematics, is based directly on the work of Thomas (1981) and is included for completeness. The serial manipulator, in essence an open loop kinematic chain consisting of a series of rigid links joined by one-degree of freedom lower-pair connectors, is illustrated in Fig. 2-1. This treatment specifically addresses only revolute (R) and prismatic (P) pairs because most other typically encountered joints can be considered as combinations of the two (e.g., a cylindric joint is analyzed as an R-P combination). Investigation of Fig. 2.1 shows that both types of joints, revolute and prismatic, have two independent parameters associated with them, (s_{jj}, Θ_j) and (s_j, Θ_{jj}), respectively. The parameter s_{jj} (or s_j) is the offset distance along the joint axis, $\underline{s}^j$, between the two links connected by the joint connects. The parameter Θ_j (or

Θ_{jj}) is the relative rotation about $\underline{s}^j$ between these links. Here the double subscript serves to indicate that the quantity is a fixed independent parameter, while the single subscript implies that the parameter is an independent variable. The two parameters a_{jk}, the fixed distance between joints j and k as measured along their common perpendicular $\underline{a}^{jk}$, and α_{jk}, the twist angle between the joint axes measured in a right-hand sense about $\underline{a}^{jk}$, define link jk.

The global reference is a fixed Cartesian system (X,Y,Z) with the Z axis directed along the first joint axis, $\underline{s}^1$, and the X axis located arbitrarily in the plane perpendicular to $\underline{s}^1$ (Note that the independent parameters associated with joint one are measured with respect to the X axis). Local (or body-fixed) dextral Cartesian reference systems ($^{(j)}x$, $^{(j)}y$, $^{(j)}z$), with $^{(j)}x$ along $\underline{a}^{jk}$ and $^{(j)}z$ along $\underline{s}^j$, are assigned for each link (jk). The vectors $\underline{a}^{jk}$ and $\underline{s}^j$ are unit vectors expressed in terms of direction cosines and given by

$$\begin{aligned} {}^{*}\underline{a}^{jk} &= T^j\,{}^{(j)}\underline{a}^{jk} = ({}^{*}X^{jk},\ {}^{*}Y^{jk},\ {}^{*}Z^{jk}) \\ {}^{*}\underline{s}^{j} &= T^j\,{}^{(j)}\underline{s}^{j} = ({}^{*}X^{j},\ {}^{*}Y^{j},\ {}^{*}Z^{j}) \end{aligned} \tag{2-45}$$

with

$$\begin{aligned} {}^{(j)}\underline{a}^{jk} &= (1,\ 0,\ 0)^T \\ {}^{(j)}\underline{s}^{j} &= (0,\ 0,\ 1)^T \end{aligned} \tag{2-46}$$

where the rotational transformation matrix T^j is

$$T^j = [\,{}^{*}\underline{a}^{jk}\ \ \underline{s}^j x\,{}^{*}\underline{a}^{jk}\ \ {}^{*}\underline{s}^j\,] \tag{2-47}$$

The preceding superscript, (j), is used to indicate that the vector is expressed in terms of the j^{th} local reference, and the preceding superscript (*) denotes a vector given in

terms of its direction cosines. The direction cosine representation of the vectors $\underline{a}^{jk}$ and $\underline{s}^{j}$ can be obtained, where all joints are assumed to be revolutes, from the initial direction cosines

$$\begin{aligned} {}^{*}\underline{s}^{1} &= (0,\ 0,\ 1)^{T} \\ {}^{*}\underline{a}^{12} &= (c\Theta_1,\ s\Theta_1,\ 0)^{T} \end{aligned} \tag{2-48}$$

and the recursive relations

$$\begin{aligned} {}^{*}\underline{s}^{j} &= T^{j-1}\,{}^{(j-1)}\underline{s}^{j} = T^{j-1}(0, -s\alpha_{(j-1)j},\ c\alpha_{(j-1)j})^{T} \\ & \qquad j = 2,3,\ldots,M \\ {}^{*}\underline{a}^{jk} &= T^{j-1}\,{}^{(j-1)}\underline{a}^{jk} = T^{j-1}(c\Theta_j, c\alpha_{(j-1)j}s\Theta_j, s\alpha_{(j-1)j}s\Theta_j)^{T} \end{aligned} \tag{2-49}$$

where $c\Theta = \cos\Theta$, $s\Theta = \sin\Theta$, etc.
If the joint is prismatic, simply replace Θ_l by Θ_{ll} and s_{ll} by s_l here and in the subsequent general equations. Finally, the position vector, $\underline{R}^{j}$, locating the origin of the j^{th} reference is given by

$$\underline{R}^{j} = s_{11}\underline{s}^{1} + \sum_{l=2}^{j}(a_{(l-1)l}\underline{a}^{(l-1)l} + s_{ll}\underline{s}^{l}) \tag{2-50}$$

Using equations (2-48), (2-49) and (2-50), the configuration of the serial manipulator is completely defined once all the zero-th order independent parameters are specified (e.g., s_{ll}, $a_{(l-1)l}$, $\alpha_{(l-1)l}$, Θ_l, etc.). The reverse position solution, that of solving for the variable input parameters given the desired hand location (e.g., $\underline{R}^{6}$, $\underline{a}^{67}$, $\underline{s}^{6}$), is (in general) considerably more difficult (Duffy, 1980) and is not addressed here. The notation involved in the kinematic

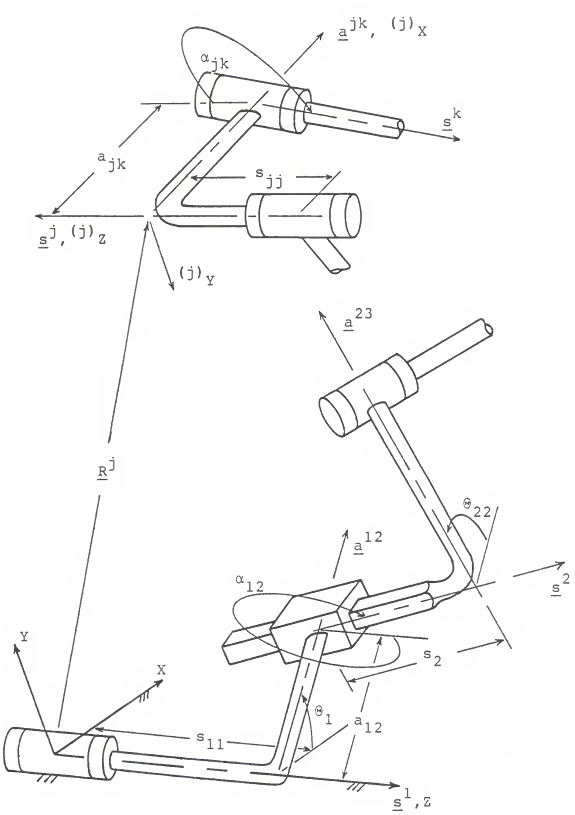


Figure 2-1. Kinematic representation of the serial manipulator

Table 2-1. Manipulator Joint and Link Parameters

$\underline{S}^j$ - Vector for Joint Axes Between Links

S_{jj} is Fixed Offset Value (Revolute)

θ_{jj} is Fixed Rotation Value (Prismatic)

φ_j - Generic Input for Joint Axis $\underline{S}^j$

S_j - Sliding Along $\underline{S}^j$

θ_j - Rotation About $\underline{S}^j$

$\left.\begin{array}{l} a_{jk} \\ \alpha_{jk} \end{array}\right\}$ - Fixed Link Dimensions Between Axes j and k

a_{jk} - Common Perpendicular Along $\underline{a}^{jk}$

α_{jk} - Twist Angle

Table 2-2. Coordinate Systems

GLOBAL SYSTEM

$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ - Absolute Reference System (Z Along Axis $\underline{S}^1$)

$\begin{pmatrix} x^j \\ y^j \\ z^j \end{pmatrix}$ - Direction Cosines for Joint $\underline{S}^j$

$\begin{pmatrix} {}^*x^{jk} \\ {}^*y^{jk} \\ {}^*z^{jk} \end{pmatrix}$ - Direction Cosines for Link $\underline{a}^{jk}$

LOCAL (j) SYSTEM

$\begin{pmatrix} {}^{(j)}x \\ {}^{(j)}y \\ {}^{(j)}z \end{pmatrix}$ - Body Fixed System for Link jk

${}^{(j)}x$ along $\underline{a}^{jk}$

${}^{(j)}z$ along $\underline{S}^{jk}$

representation of the serial manipulator is given in Tables 2.1 and 2.2 for quick reference.

First-order kinematics

In the development of the generalized kinematics, the specific nature of the dependent motion parameters (u^p) and the independent input parameters (Θ_n) were not taken into account. In the case of the serial manipulator there is a fundamental difference in the nature of the two basic motion parameters considered. In this treatment the two parameters are the cartesian-referenced link orientations (which are not considered true coordinates) and the cartesian-referenced coordinates of a point (which are true coordinates). Also, the effect of an input on these parameters depends on whether the input is a revolute or prismatic joint. In light of these facts, the kinematics for rotational parameters amd those for translational parameters are treated separately, and the results for both revolute and prismatic inputs are given in each case.

In dealing with the kinematics of rotational parameters, it is not convenient to start with an equation of the form of equation (2-1) (i.e., $\underline{u} = f(\underline{\varphi})$) since finite angles of rotation cannot be represented by vectors. Fortunately, however, it is possible to represent infinitesimal rotations by vectors (Meirovitch, 1970), and these infinitesimal rotations can most readily be interpreted as the angular velocity ($\underline{\omega}$) of the body. So, instead of using a zeroth-order holonomic constraint equation (e.g., $\underline{u} = f(\underline{\varphi})$)

to derive a link's first-order rotational influence coefficients, a first-order non-holonomic constraint equation expressing the link's angular velocity ($\underline{\omega}^{jk}$) is used.

Referring to Fig. 2-1, the angular velocity addition theorem gives the angular velocity ($\underline{\omega}^{jk}$) of link jk as simply the sum of the relative angular velocities between preceding links in the chain

$$\underline{\omega}^{jk} = \sum_{n=1}^{j} \dot{\Theta}_n \, \underline{s}^n, \; n = 1,2,\ldots,L \tag{2-51}$$

where, $\dot{\Theta}_n \underline{s}^n$ is the relative angular velocity between links (n-1)n and n(n+1), and $\dot{\Theta}_n$ is identically zero for a prismatic joint. Investigation of equation (2-51) shows that the angular velocity ($\underline{\omega}^{jk}$) can be separated into a function of position (i.e., $\underline{s}^n = f(\underline{\varphi})$ operated on by a function of time (i.e., $\underline{\Theta} = \underline{\Theta}(t)$) or, in the form of equation (2-7),

$$\underline{\omega}^{jk} = f(\underline{\varphi})*\Theta(t) \equiv [G_{\varphi}^{jk}]\dot{\underline{\varphi}} \tag{2-52}$$

Now, in order to obtain the 3xM configuration dependent matrix $[G_{\varphi}^{jk}]$, one has that (also see equation (2-12))

$$\begin{aligned} \frac{\partial(\underline{\omega}^{jk})}{\partial\dot{\underline{\varphi}}} &= \frac{\partial([G_{\varphi}^{jk}]\dot{\underline{\varphi}})}{\partial\dot{\underline{\varphi}}} \\ &= [G_{\varphi}^{jk}]\frac{\partial(\dot{\underline{\varphi}})}{\partial\dot{\underline{\varphi}}} \\ &= [G_{\varphi}^{jk}] \end{aligned} \tag{2-53}$$

In fact, for any position dependent vector ($\underline{u} = \underline{u}(\underline{\varphi})$) one has from calculus that

$$\frac{\partial}{\dot{\Phi}_n}\left(\frac{d(\underline{u})}{dt}\right) = \frac{\partial}{\partial\dot{\Phi}_n}\left(\frac{\partial(\underline{u})}{\partial\underline{\varphi}}\frac{d\underline{\varphi}}{dt}\right)$$
$$= \frac{\partial(\underline{u})}{\partial\underline{\varphi}}\ \frac{\partial(\dot{\underline{\varphi}})}{\partial\dot{\Phi}_n}$$
$$= \left[\frac{\partial(\underline{u})}{\partial\underline{\varphi}}\right]_{;n} \tag{2-54}$$
$$= \frac{\partial(\underline{u})}{\partial\underline{\varphi}}$$

This turns out to be a very convenient relationship and is used to obtain the kinematic influence coefficients for translational parameters even though it is not necessary since the position vector $\underline{u} = \underline{P} = (X^P, Y^P, Z^P)^T$ exists and could be differentiated directly.

Returning to the question at hand, from equations (2-51), (2-52) and (2-53), the rate of change of the angular orientation of link jk is

$$\underline{\omega}^{jk} = [G_{\varphi}^{jk}]\dot{\underline{\varphi}} \tag{2-55}$$

where

$$\underline{g}_n^{jk} = \frac{\partial(\underline{\omega}^{jk})}{\partial\dot{\Phi}_n} = \frac{\partial}{\partial\dot{\Phi}_n}\left(\sum_{i=1}^{j}\dot{\Theta}_i\ \underline{s}^i\right) = \begin{cases} \underline{s}^n, & n \le j;\ \varphi_n = \Theta_n \text{ (revolute)} \\ \underline{o}\ , & \text{otherwise} \end{cases} \tag{2-56}$$

Here, the angular velocities (e.g., $\underline{\omega}^{jk}$) are expressed in terms of cartesian reference frames and denoted, in a component sense, as

$$\underline{\omega}^{jk} = (\omega^{jx}, \omega^{jy}, \omega^{jz})^T \tag{2-57}$$

Consistent with this notation, the influence coefficients take the form

$$g_n^{jk} = (g_n^{jx},\ g_n^{jy},\ g_n^{jz})^T \tag{2-58}$$

where

$$\underline{s}^n = (*X^n,\ *Y^n,\ *Z^n)^T \tag{2-59}$$

The translational velocity ($\dot{\underline{P}}$) of a point ($\underline{P}$) in link jk, and hence (from equation (2-54)) the translation influence coefficients, can be derived from

$$\underline{P} = \underline{R}^j + T^{j\,(j)}\underline{P} \tag{2-60}$$

where, ${}^{(j)}\underline{P}$ is the location of point P in terms of the body fixed reference j, and multiplication of ${}^{(j)}\underline{P}$ by the rotation matrix T^j transforms the coordinates from the j^{th} reference to an intermediate system with the same origin but with axes parallel to the global reference (X, Y, Z). Substituting equation (2-50) for $\underline{R}^j$ and differentiating equation (2-60) gives the velocity of point P in link jk as

$$\dot{\underline{P}} = \dot{s}_1\underline{s}^1 + \sum_{l=2}^{j} (a_{(l-1)l}\dot{\underline{a}}^{(l-1)l} + s_{ll}\dot{\underline{s}}^l + \dot{s}_l\underline{s}^l) + \frac{d}{dt}(T^{j\,(j)}\underline{P}) \tag{2-61}$$

Now, since $\underline{a}^{(l-1)l}$ and $\underline{s}^l$ are unit vectors fixed in link (l-1)l and $(T^{j\,(j)}\underline{P})$ is a vector fixed in link jk, the time rate of change of these vectors can be expressed in terms of the vector cross product as

$$\dot{\underline{a}}^{(l-1)l} = \underline{\omega}^{(l-1)l} \times \underline{a}^{(l-1)l} = \left(\sum_{n=1}^{l-1} \dot{\Theta}_n\ \underline{s}^n\right) \times \underline{a}^{(l-1)l}$$

$$\dot{\underline{s}}^l = \underline{\omega}^{(l-1)l} \times \underline{s}^l = \left(\sum_{n=1}^{l-1} \dot{\Theta}_n\ \underline{s}^n\right) \times \underline{s}^l \tag{2-62}$$

$$\frac{d}{dt}(T^{j(j)}\underline{P}) = \underline{\omega}^{jk} \times (T^{j(j)}\underline{P}) = \left(\sum_{n=1}^{l-1} \dot{\Theta}_n\, \underline{s}^n\right) \times (T^{j(j)}\underline{P})$$

Substitution of equations (2-62) into equation (2-61) and regrouping of terms leads to

$$\begin{aligned}\dot{\underline{P}} &= \sum_{n=1}^{j} \{\dot{s}_n\underline{s}^n + \dot{\Theta}_n\underline{s}^n \times [\sum_{l=n+1}^{j}(a_{(l-1)l}\underline{a}^{(l-1)l} + s_{ll}\underline{s}^l) \\ &\qquad + T^{j(j)}\underline{P}]\} \\ &= \sum_{n=1}^{j}\{\dot{s}_n\underline{s}^n + \dot{\Theta}_n\underline{s}^n \times (\underline{P}-\underline{R}^n)\}\end{aligned} \tag{2-63}$$

where, referring to Fig. (2-1),

$$\sum_{l=n+1}^{j}(a_{(l-1)l}\underline{a}^{(l-1)l} + s_{ll}\underline{s}^l) + T^{j(j)}\underline{P} = (\underline{P}-\underline{R}_n) \tag{2-64}$$

is a vector from the origin of the n^{th} reference to the point P. Finally, recalling equation (2-54), the velocity of point P can be expressed in the form

$$\dot{\underline{P}} = [G_{\varphi}^{P}]\dot{\underline{\varphi}} \tag{2-65}$$

where

$$\underline{g}_n^P = \frac{\partial \underline{P}}{\partial \varphi_n} = \begin{cases} \underline{s}^n \times (\underline{P}-\underline{R}_n) & , n\le j;\ \varphi_n=\Theta_n \text{ (revolute)} \\ \underline{s}^n & , n\le j;\ \varphi_n=s_n \text{ (prismatic)} \\ \underline{o} & , n>j \end{cases} \tag{2-66}$$

<u>Second-order kinematics</u>

Now, relying heavily on the development presented for the second- and third-order kinematics of a general motion

parameter, the higher-order kinematic influence coefficients, and hence the higher-order kinematics for the serial manipulator, can be obtained by directly operating on the first-order coefficients given in equations (2-27) and (2-66). Observing that both the rotational and translational influence coefficients are expressed in terms of position dependent vectors leads one to again use the relationship of equation (2-54) to obtain the (h,H) and (d,D) functions.

Recalling the first-order rotational coefficients and equations (2-54), one has

$$\frac{\underline{d}}{dt}(g^{jk}_{n}) = \begin{cases} \underline{s}^{n}, & n \le j \ ; \ \varphi_n = \Theta_n \\ \underline{o}, & \text{otherwise} \end{cases} \qquad (2\text{-}67)$$

where

$$\underline{s}_n = \left(\sum_{i=1}^{n-1} \Theta_i \, \underline{s}^{i} \right) \times \underline{s}^{n} \qquad (2\text{-}68)$$

Performing the partial differentiation with respect to the m^{th} generalized velocity yields, for the second-order rotational coefficients, the expected non-symmetric (i.e., $\underline{h}^{jk}_{mn} = \underline{h}^{jk}_{nm}$) result

$$\underline{h}^{jk}_{mn} \equiv \frac{\partial}{\partial \varphi_m}(\underline{g}^{jk}_{n}) = \frac{\partial}{\partial \varphi_m}(\underline{g}^{jk}_{n}) = \begin{cases} \underline{s}^{m} \times \underline{s}^{n}, & m < n \le j \ ; \ \varphi_m = \Theta_m, \ \varphi_n = \Theta_n \\ \underline{o}, & \text{otherwise} \end{cases} \qquad (2\text{-}69)$$

Also, notice that the rotational (h,H) functions are zero if either input is prismatic. Referring to the general form of equation (2-27), the angular acceleration of link jk becomes

$$\underline{\alpha}^{jk} = [G^{jk}_{\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]\dot{\underline{\varphi}} \tag{2-70}$$

where

$$\underline{\alpha}^{jk} = (\alpha^{jx},\ \alpha^{jy},\ \alpha^{jz})^T \tag{2-71}$$

and

$$[H^{jk}_{\varphi\varphi}] = 3 \times M \times M \tag{2-72}$$

with

$$[H^{jk}_{\varphi\varphi}]_{;m;n} = \underline{h}^{jk}_{mn} = (h^{jx}_{mn},\ h^{jy}_{mn},\ h^{jz}_{mn})^T \tag{2-73}$$

The second order kinematics of a point P in link jk are perhaps most easily understood by identifying all possible input combinations and addressing each situation independently. Looking first at the case where the n^{th} generalized coordinate is a prismatic joint (i.e., $\varphi_n = s_n$), the time rate of change of the (g,G) function is

$$\dot{\underline{g}}^P_n = \dot{\underline{s}}^n \ ,\ n \le j\ ;\ \varphi_n = s_n \tag{2-74}$$

Since $\underline{s}^n$ can be viewed as a unit vector fixed in link (n-1)n, one has that

$$\dot{\underline{g}}^P_n = \underline{\omega}^{(n-1)n} \times \underline{s}^n = \left(\sum_{i=1}^{n-1} \dot{\theta}_i\ \underline{s}^i\right) \times \underline{s}^n \ ,\ n \le j;\ \varphi_n = s_n \tag{2-75}$$

Taking the partial derivative of this expression yields, for a prismatic joint n

$$\underline{h}^P_{mn} = \frac{\partial}{\partial \dot{\phi}_m}(\underline{g}^P_n) = \begin{cases} \underline{s}^m \times \underline{s}^n , & m<n\leq j \ ; \ \varphi_m=\Theta_m, \ \varphi_n=s_n \\ \underline{o} , & n\leq m\leq j \ ; \ \varphi_m=\Theta_m, \ \varphi_n=s_n \\ \underline{o} , & \varphi_m=s_m \ ; \ \varphi_n=s_n \end{cases} \qquad (2\text{-}76)$$

Now, considering the case where the n^{th} joint is a revolute (i.e., $\varphi_n = \Theta_n$), the time rate of change of the influence coefficient is

$$\dot{\underline{g}}^P_n = \dot{\underline{s}}^n \times (\underline{P}-\underline{R}^n) + \underline{s}^n \times (\dot{\underline{P}}-\dot{\underline{R}}^n) \qquad (2\text{-}77)$$

or, by differentiating equation (2-64) to obtain $(\underline{P}-\underline{R}^n)$ and substituting the cross product form of equations (2-62) for the vector derivatives,

$$\begin{aligned} \dot{\underline{g}}^P_n = & \left(\left(\sum_{i=1}^{n-1} \dot{\Theta}_i \, \underline{s}^i\right) \times \underline{s}^n\right) \times (\underline{P}-\underline{R}^n) \\ & + \underline{s}^n \times \left(\sum_{l=n+1}^{j}\left[\left(\sum_{i=1}^{l-1}\dot{\Theta}_i \, \underline{s}^i\right) \times (a_{(l-1)l}\underline{a}^{(l-1)l} + s_{ll}\underline{s}^l)\right]\right. \\ & \left. + \left(\sum_{i=1}^{j} \dot{\Theta}_i \, \underline{s}^i\right) \times (T^{j(j)}\underline{P})\right) \\ & + \underline{s}^n \times \left(\sum_{l=n+1}^{j} \dot{s}_l \, \underline{s}^l\right) \end{aligned} \qquad (2\text{-}78)$$

Investigating the case where the m^{th} input is a prismatic joint (i.e., $\varphi_m = s_m$), one sees immediately from the last term of the preceding equation that

$$\underline{h}^P_{mn} = \frac{\partial \dot{\underline{g}}^P_n}{\partial \dot{s}_m} = \begin{cases} \underline{s}^n \times \underline{s}^m , & n<m\leq j \ ; \ \varphi_n=\Theta_n, \ \varphi_m=s_m \\ \underline{o} , & m\leq n\leq j \end{cases} \qquad (2\text{-}79)$$

The reader should note that this equation is consistent with equation (2-76) and shows that the result is non-zero only if the revolute precedes the prismatic joint in the serial chain. The symmetry (i.e., $\underline{h}^P_{mn} = \underline{h}^P_{nm}$) of the translational (h,H) functions should also be noted. Now, when the m^{th} input is also a revolute (i.e., $\varphi_m = \Theta_m$) it is best to look at the case in two steps. First, when m ≥ n, from the second term of equation (2-78)

$$\frac{\partial \underline{\dot{g}}^P_n}{\partial \Theta_m} = \underline{s}^n \times (\underline{s}^m \times [\sum_{l=m+1}^{j}(a_{(l-1)l}\underline{a}^{(l-1)l} + s_{ll}\underline{s}^l) + T^{j\,(j)}\underline{P})) \tag{2-80}$$

or, from equation (2-64)

$$\frac{\partial \underline{\dot{g}}^P_n}{\partial \Theta_m} = \underline{s}^n \times (\underline{s}^m \times (\underline{P}-\underline{R}^m)) \quad , \; n<m\leq j \tag{2-81}$$

If m < n, both of the first two terms in equation (2-78) are involved and, one finds that

$$\frac{\partial \underline{\dot{g}}^P_n}{\partial \Theta_m} = (\underline{s}^m \times \underline{s}^n) \times (\underline{P}-\underline{R}^n) \tag{2-82}$$
$$+ \underline{s}^n \times (\underline{s}^m \times (\sum_{l=n+1}^{j}(a_{(l-1)l}\underline{a}^{(l-1)l} + s_{ll}\underline{s}^l + T^{j\,(j)}\underline{P}))$$

or, again recalling equation (2-64)

$$\frac{\partial \underline{\dot{g}}^P_n}{\partial \Theta_m} = (\underline{s}^m \times \underline{s}^n) \times (\underline{P}-\underline{R}^n) + \underline{s}^n \times (\underline{s}^m \times (\underline{P}-\underline{R}^n))$$
$$= \underline{s}^m \times (\underline{s}^n \times (\underline{P}-\underline{R}^n)) \quad , \; m\leq n\leq j \tag{2-83}$$

Combining this result with that of equation (2-81) further illustrates the symmetry inherent in the translational (h,H) functions and yields

$$\underline{h}^{P}_{mn} = \underline{h}^{P}_{mn} = \underline{s}^{i} \times (\underline{s}^{j} \times (\underline{P}-\underline{R}^{j})) \quad , \; i = \min\,(m,n)$$
$$j = \max\,(m,n) \qquad (2\text{-}84)$$

Before proceeding with the kinematic development, it will prove beneficial to reinvestigate a representative sampling of the first- and second-order translational influence coefficients in a more geometric light. Recalling equations (2-8) and (2-66), the first-order geometric influence coefficient for a point P in link jk with respect to the n^{th} input is a vector ($\underline{g}^{P}_{n}$) which when multiplied by the n^{th} generalized velocity ($\dot{\phi}_n$) yields the n^{th} partial velocity ($\underline{\dot{P}}_n$) of the vector ($\underline{P}$)

$$\underline{\dot{P}} = \sum_{n=1}^{j} \underline{\dot{P}}_n = \sum_{n=1}^{j} \underline{g}^{P}_{n}\, \dot{\phi}_n \; , \; \underline{g}^{P}_{n} = 0 \;\; ; \; n>j \qquad (2\text{-}85)$$

Let us first investigate the case where the n^{th} input is a prismatic joint (i.e., $\varphi_n = s_n$) and the desire is to find $({}_s\underline{g}^{P}_{n})$. Here, the preceding subscript (s) is introduced to indicate that input n is prismatic. Referring to Fig. 2-2, one can readily see that the vector ($\underline{P}$), and hence, its velocity ($\underline{\dot{P}}$), can be expressed in a variety of ways. In the manipulator of Fig. 2-2, where the second joint is prismatic (i.e., $\varphi_2 = s_2$), it is advantageous to view the vector ($\underline{P}$) as

$$\underline{P} = \underline{R}^{2} + (\underline{P}-\underline{R}^{2}) \qquad (2\text{-}86)$$

Here the quantity $(\underline{P}-\underline{R}^2)$ is considered a free vector and the pure translation of a free vector in no way alters the vector (i.e., $(\dot{\underline{P}-\underline{R}^2})=0$). With this representation it is readily apparent that the action of the slider (s_2) only affects the vector $(\underline{R}^2)$. Also, by inspection, the rate of change of this vector $(\underline{R}^2)$, and therefore, the rate of change of $\underline{P}$ due to the action of s_2, is

$$\dot{\underline{P}}_2 = \dot{\underline{R}}^2_2 = \dot{s}_2\underline{s}^2 \tag{2-87}$$

since

$$\underline{R}^2 = s_{11}\underline{s}^1 + a_{12}\underline{a}^{12} + s_2\underline{s}^2 \tag{2-88}$$

From equation (2-77) one obtains the expected result (see equation (2-66))

$${}_s\underline{g}_n^P = \underline{s}^n \ , \ n=2 \tag{2-89}$$

Now, consider the derivation of the (g,G) function for a revolute joint (e.g., $\varphi_n = \Theta_3$). In this situation, the coefficient $({}_\Theta\underline{g}_3^P)$ is most easily found by expressing the position vector $\underline{P}$ as

$$\underline{P} = \underline{R}^3 + (\underline{P}-\underline{R}^3) \tag{2-90}$$

In this case it is the vector $\underline{R}^3$ that is unaffected by the action of the revolute (Θ_3). Recalling the vector cross product representation of the velocity of a body fixed vector, one has the partial velocity of $\underline{P}$ due to the third input as

$$\dot{\underline{P}}_3 = (\dot{\underline{P}-\underline{R}^3})_3 = \dot{\Theta}_3\underline{s}^3 \text{ x } (\underline{P}-\underline{R}^3) \tag{2-91}$$

since $(\underline{P}-\underline{R}^3)$ can be viewed as a vector fixed in link 34. This again yields the expected result (see equation (2-66))

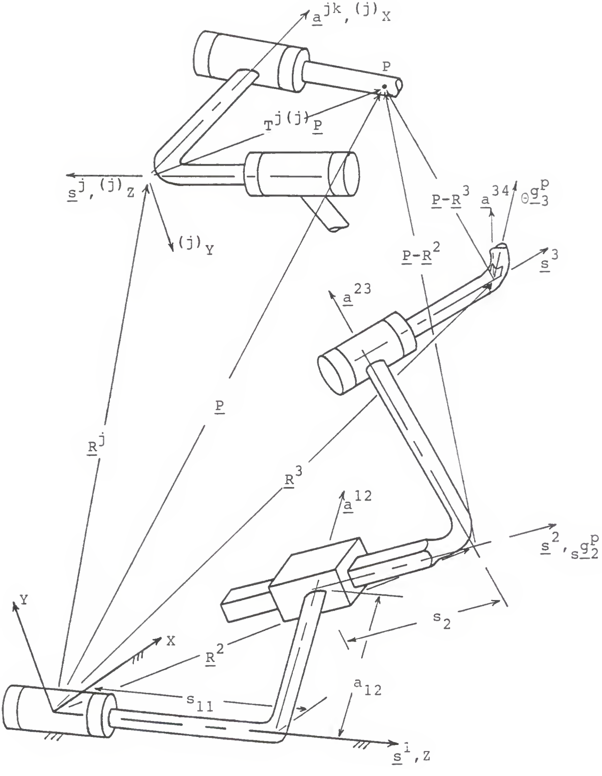


Figure 2-2. First-order kinematic influence coefficients

$${}_{\Theta}g_{n}^{P} = \underline{s}^{n} \times (\underline{P}-\underline{R}^{n}) \ , \ n=3 \qquad (2\text{-}92)$$

The difference between the translational first-order coefficients for revolute and prismatic joints is perhaps more easily understood now, from equations (2-86) through (2-92), than from the more analytical development resulting in equation (2-66). That is, by expressing the vector ($\underline{P}$) as

$$\underline{P} = \underline{R}^{n} + (\underline{P}-\underline{R}^{n}) \qquad (2\text{-}93)$$

it becomes obvious that, if the n^{th} input is prismatic (i.e., $\Theta_n = s_n$) the location of the origin of the local reference ($\underline{R}^n$) fixed to link n(n+1) is translated and, if the n^{th} input is a revolute (i.e., $\varphi_n = \Theta_n$) the vector ($\underline{P}-\underline{R}^n$) from the local origin to the point of interest is rotated.

To reinterpret the second-order coefficients (i.e., $\underline{h}_{mn}^{P}$), take the case where the n^{th} joint is a revolute. Here, referring to Fig. 2-3, one has

$${}_{\Theta}\underline{g}_{n}^{P} = \underline{s}^{n} \times (\underline{P}- \underline{R}^{n}) \qquad (2\text{-}94)$$

First, if m ≤ n, one can obtain ($\underline{h}_{mn}^{P}$) immediately by noting that (${}_{\Theta}\underline{g}_{n}^{P}$) as expressed in equation (2-94) represents a vector fixed in link n(n+1). From the first-order results it follows that

$${}_{s\Theta}\underline{h}_{mn}^{P} = \underline{o} \ , \ m \le n \le j \qquad (2\text{-}95)$$

and

$${}_{\Theta\Theta}\underline{h}_{mn}^{P} = \underline{s}^{m} \times (\underline{s}^{n} \times (\underline{P}-\underline{R}^{n})) \ , \ m \le n \le j \qquad (2\text{-}96)$$

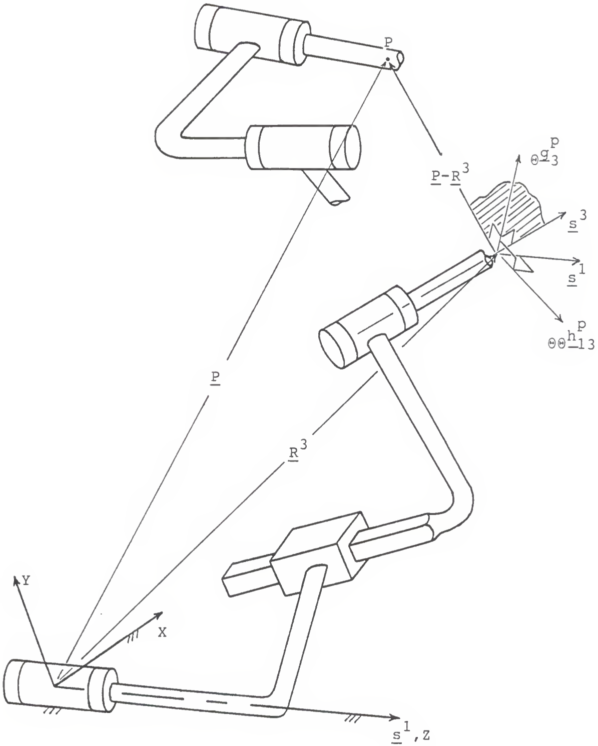


Figure 2-3. Second-order kinematic influence coefficients when second joint is a revolute

since $({}_{\Theta}g_n^P)$ is affected in the same manner as $(\underline{P}-\underline{R}^n)$ was previously. Now, if m > n, the first step is to reexpress equation (2-94) as (see Fig. 2-4)

$$ {}_{\Theta}g_n^P = \underline{s}^n \times [(\underline{P}-\underline{R}^m) + (\underline{R}^m-\underline{R}^n)] \qquad (2\text{-}97)$$

Here, from past results, it can be seen that if the m^{th} input is a prismatic joint only the vector $(\underline{R}^m-\underline{R}^n)$ is effected. On the other hand, if the m^{th} input is a revolute the only affect is on the vector $(\underline{P}-\underline{R}^m)$. This gives the second-order results as

$$ {}_{s\Theta}\underline{h}_{mn}^P = \underline{s}^n \times \underline{s}^m \ , \ n<m\leq j \qquad (2\text{-}98)$$

and

$$ {}_{\Theta\Theta}\underline{h}_{mn}^P = \underline{s}^n \times (\underline{s}^m \times (\underline{P}-\underline{R}^m)) \ , \ n<m\leq j \qquad (2\text{-}99)$$

The remaining second-order influence coefficients are left for the reader to reinterpret and compare with the more analytically derived equations (2-76), (2-79), (2-81) and (2-83).

Now, with the second-order influence coefficients fully established, the acceleration of point (P) can be obtained as

$$ \ddot{\underline{P}} = [G_{\phi}^P]\ddot{\underline{\phi}} + \dot{\underline{\phi}}^T[H_{\phi\phi}^P]\dot{\underline{\phi}} \qquad (2\text{-}100)$$

where

$$ \ddot{\underline{P}} = (\ddot{X}^P, \ \ddot{Y}^P, \ \ddot{Z}^P)^T \qquad (2\text{-}101)$$

and

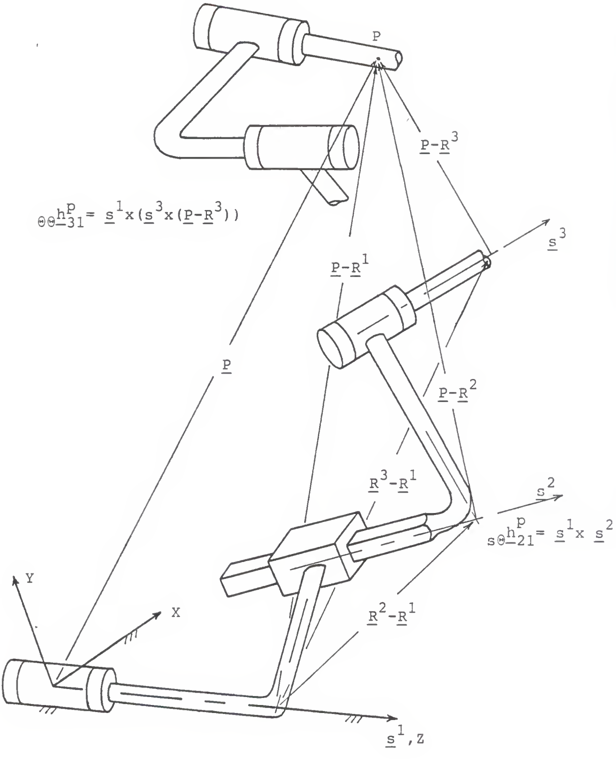


Figure 2-4. Second-order kinematic influence coefficients when first joint is a revolute.

$$[H^{P}_{\varphi\varphi}] = 3 \times M \times M \tag{2-102}$$

with

$$[H^{P}_{\varphi\varphi}]_{;m;n} = [H^{P}_{\varphi\varphi}]_{;n;m} = \underline{h}^{P}_{mn} = (h^{PX}_{mn}, h^{PY}_{mn}, h^{PZ}_{mn})^{T} \tag{2-103}$$

<u>Third-order kinematics</u>

The third-order kinematics require the development of the third-order influence coefficients (i.e., the (d,D) functions). Here, only a few representative cases will be derived. The rotational coefficients are obtained via the strict analytic approach, whereas the translational coefficients are addressed using the more illustrative graphic approach. The complete third-order coefficient results, along with those for the first- and second-order coefficients are presented in Tables 2-3, 2-4 and 2-5.

Again, the rotational influence coefficients will be investigated first. Here, if any of the three inputs involved is a prismatic joint, then the third-order geometric derivative is identically zero. This result is due to the fact that the translation of a free vector does not alter the vector (as was also the case for the (h,H) function). Now, if all inputs considered are revolutes, one has

$$_{\Theta\Theta}\underline{h}^{jk}_{mn} = \begin{cases} \underline{s}^{m} \times \underline{s}^{n} & , m<n\le j \\ \underline{o} & , n\le m \end{cases} \tag{2-104}$$

and the concern is to determine the effect of the third revolute ($\varphi_l = \Theta_l$) on this second-order property. There are two unique non-zero situations here, $l<m<n$ and $m \le l < n$. If $l<m<n$, then $\underline{h}^{jk}_{mn}$ can be considered as a vector fixed in link $(m-1)m$ and the third-order geometric derivative is

$$\begin{aligned} {}_{\Theta\Theta\Theta}\underline{d}^{jk}_{lmn} &= \frac{\partial}{\partial \dot{\varphi}_l}\Big(\Big(\sum_{i=1}^{m-1} \dot{\Theta}_i\, \underline{s}^i\Big) \times (\underline{s}^m \times \underline{s}^n)\Big) \\ &= \underline{s}^l \times (\underline{s}^m \times \underline{s}^n)\ , \quad l<m<m\le j \end{aligned} \tag{2-105}$$

If $m \le l < n$, then $\underline{s}^m$ is unaffected by the l^{th} input and $\underline{s}^n$ can be considered as a vector fixed in link $(n-1)n$ yielding

$$\begin{aligned} {}_{\Theta\Theta\Theta}\underline{d}^{jk}_{lmn} &= \underline{s}^m \times \frac{\partial}{\partial \varphi_l}\Big(\Big(\sum_{i=1}^{n-1} \dot{\Theta}_i\, \underline{s}^i\Big) \times \underline{s}^n\Big) \\ &= \underline{s}^m \times (\underline{s}^l \times \underline{s}^n)\ , \quad m \le l < n \le j \end{aligned} \tag{2-106}$$

One can now write the time rate of change of the angular acceleration of link j^k as (see equation (2-44))

$$\begin{aligned} \dot{\underline{\alpha}}^{jk} &= [G^{jk}_{\varphi}]\dddot{\underline{\varphi}} + \ddot{\underline{\varphi}}^T[[H^{jk}_{\varphi\varphi}] + 2[H^{jk}_{\varphi\varphi}]^T]\dot{\underline{\varphi}} \\ &\quad + (\dot{\underline{\varphi}}^T[D^{jk}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}})\dot{\underline{\varphi}} \end{aligned} \tag{2-107}$$

where all algebraic operations are as defined previously in the general kinematics section and

$$[D^{jk}_{\varphi\varphi\varphi}] = 3 \times M \times M \times M \tag{2-108}$$

with

$$[D^{jk}_{\varphi\varphi\varphi}]_{;l;m;n} = \underline{d}^{jk}_{lmn} = (d^{jX}_{lmn}, d^{jY}_{lmn}, d^{jZ}_{lmn})^T \tag{2-109}$$

The third-order translational coefficients where one of the inputs is prismatic can be obtained in exactly the same manner as were the rotational coefficients for three revolute inputs (since the symmetry exhibited by the second-order translational coefficients carries over to the higher order properties), where

$$\frac{\partial}{\partial s_n}\left(\frac{\partial}{\partial \Theta_l}\left(\frac{\partial \underline{P}}{\partial \Theta_m}\right)\right) = \frac{\partial}{\partial \Theta_l}\left(\frac{\partial}{\partial \Theta_m}\left(\frac{\partial \underline{P}}{\partial s_n}\right)\right) = \ldots \tag{2-110}$$

As with the translational (h,H) functions the only non-zero result occurs when the prismatic joint is the last joint considered and can be expressed as

$${}_{s\Theta\Theta}\underline{d}^{P}_{nlm} = {}_{\Theta\Theta s}\underline{d}^{P}_{lmn} = \ldots = \underline{s}^{i} \times (\underline{s}^{j} \times \underline{s}^{n}) \;,\; i=\min(l,m),\; j=\max(l,m) \tag{2-111}$$

Considering the situation where all three inputs concerned are revolutes, and $m \le n \le j$, one has

$$\underline{h}^{P}_{mn} = \underline{s}^{m} \times (\underline{s}^{n} \times (\underline{P}-\underline{R}^{n})) \tag{2-112}$$

Taking the more graphic approach (see Fig. 2-5), for $l \le m$, ($\underline{h}^{P}_{mn}$) can be considered as a vector fixed in link m(m+1), yielding

$$\underline{d}^{P}_{lmn} = \underline{s}^{l} \times (\underline{s}^{m} \times (\underline{s}^{n} \times (\underline{P}-\underline{R}^{n}))) \;,\; l \le m \le n \le j \tag{2-113}$$

For $m < l \le n$, the vector $\underline{s}^{m}$ is not effected and $(\underline{s}^{n} \times (\underline{P}-\underline{R}^{n}))$ can be viewed as a vector fixed in link n(n+1), giving

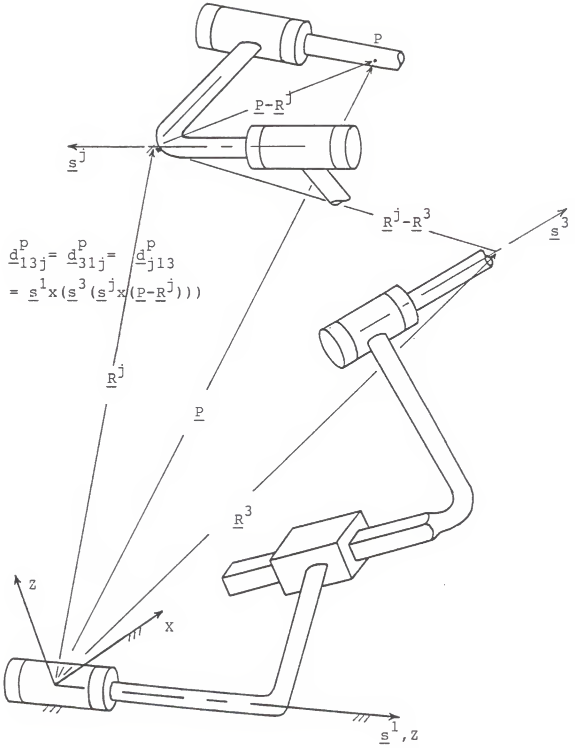


Figure 2-5. Third-order kinematic influence coefficients

$$\underline{d}^{P}_{lmn} = \underline{s}^{m} \times (\underline{s}^{l} \times (\underline{s}^{n} \times (\underline{P}-\underline{R}^{n}))) \quad , \; m<l\le n\le j \tag{2-114}$$

Finally, for m<n<l, by reexpressing $(\underline{P}-\underline{R}^{n})$ in equation (2-112) as $[(\underline{P}-\underline{R}^{l}) + (\underline{R}^{l}-\underline{R}^{n})]$ one sees that the only affected vector is $(\underline{P}-\underline{R}^{l})$, so

$$\underline{d}^{P}_{lmn} = \underline{s}^{m} \times (\underline{s}^{n} \times (\underline{s}^{l} \times (\underline{P}-\underline{R}^{l}))) \quad , \; m<n<l\le j \tag{2-115}$$

The remaining translational third-order geometric derivatives are left for the reader to derive and are given in Table 2-3. The time rate of change of the acceleration of a point P in link jk can now be expressed, from equation (2-44), as

$$\dddot{\underline{P}} = [G^{P}_{\varphi}]\dddot{\underline{\varphi}} + 3\ddot{\underline{\varphi}}^{T}[H^{P}_{\varphi\varphi}]\dot{\underline{\varphi}} + [\dot{\underline{\varphi}}^{T}[D^{P}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}}]\dot{\underline{\varphi}} \tag{2-116}$$

where

$$[D^{P}_{\varphi\varphi\varphi}] = 3 \times M \times M \times M \tag{2-117}$$

with

$$[D^{P}_{\varphi\varphi\varphi}]_{;l;m;n} = \underline{d}^{P}_{lmn} \tag{2-118}$$

and the symmetry of the translational (h,H) functions has been observed (i.e., $[H^{P}_{\varphi\varphi}]^{T} = [H^{P}_{\varphi\varphi}]$). This completes the treatment of the kinematics of serial manipulators.

Table 2-3. First Order Influence Coefficients for Serial Manipulators

Symbol	Joint Type at M	Joint Type at N	Restrictions	Value
Rotational	-	-	$n \le j$	$\underline{S}^n$
$[G_{\varphi}^{jk}]_n$	-	-	$n > j$	0
	-	-	All n	0
Translational	-	R	$n \le j$	$\underline{S}^n \times (\underline{P}-\underline{R}^n)$
$[{}^jG_{\varphi}^{P}]_n$	-	P	$n \le j$	$\underline{S}^n$
	-	R,P	$n > j$	0

Table 2.4 Second Order Influence Coefficients for Serial Manipulators

Symbol	Joint Type At M	Joint Type At N	Restrictions	Value
Rotational $[H^{jk}_{\varphi\varphi}]_{m;n}$	R	R	$m<n\leq j$	$\underline{S}^m \times \underline{S}^n$
	R	R	$m<n$ or $n>j$	0
	P	P	All m,n	0
	R	P	All m,n	0
Translational $[{}^jH^p_{\varphi\varphi}]_{M;n}$	R	R	$m<n\leq j$	$\underline{S}^m \times [\underline{S}^n \times (\underline{P}-\underline{R}^n)]$
	R	R	$n<m\leq j$	$\underline{S}^n \times [\underline{S}^m \times (\underline{P}-R^m)]$
	P	R	$n<m\leq j$	$\underline{S}^n \times \underline{S}^m$
	R	P	$m<n\leq j$	$\underline{S}^m \times \underline{S}^n$
	P	R	$m<n\leq j$	0
	R	P	$n<m\leq j$	0
	R,P	R,P	(m or n) $<$ j	0

Table 2-5. Third-order Influence Coefficients for Serial Manipulators

Symbol	Joint Type l	m	n	Restrictions	Value
Rotational	R	R	R	l≤m<n≤j	$\underline{S}^l x(\underline{S}^m x \underline{S}^n)$
$[D^{jk}_{\varphi\varphi\varphi}]_{l;m;n}$	R	R	R	m<l≤j	$\underline{S}^m x(\underline{S}^l x \underline{S}^n)$
	R	R	R	m,l≤n or nj	$\underline{0}$
	-	-	P	all l,m,n	$\underline{0}$
Translational	R	R	R	l≤m≤n≤j	$\underline{S}^l x(\underline{S}^m x[\underline{S}^n x(\underline{P}-\underline{R}^n)])$
$[D^{P}_{\varphi\varphi\varphi}]_{l;m;n}$	R	R	R	m<l≤n≤j	$\underline{S}^m x(\underline{S}^l x[\underline{S}^n x(\underline{P}-\underline{R}^n)])$
	R	R	R	m<n<l≤j	$\underline{S}^m x(\underline{S}^n x[\underline{S}^l x(\underline{P}-\underline{R}^l)])$
	R	R	R	n<m	by symmetry
	R	R	P	l≤m≤n≤j	$\underline{S}^l x(\underline{S}^m x \underline{S}^n)$
	R	R	P	m≤l<n≤j	$\underline{S}^m x(\underline{S}^l x \underline{S}^n)$
	R	R	P	m,l≤n or n>j	$\underline{0}$
	R	P	P	all other l,m,n≤j	by symmetry
	-	P	P	all l,m,n	$\underline{0}$
	-	-	-	l,m,n>j	$\underline{0}$

The Dynamic Model

This section deals with the determination of the generalized input loads (forces and, or torques) required to cause the system in question to undergo some arbitrary desired motion. The derivation of the describing equations presented herein is based almost entirely on two fundamental principles of mechanics (the principle of virtual work and d'Alembert's principle). Lagrange's equation could have been employed, yielding the same results (typically in scalar form as in Thomas, 1981). However, it is felt that the approach taken here more directly stresses the geometric nature of the result.

The principle of virtual work is used to obtain the generalized forces (or torques) necessary to counteract externally applied loads and put the system in a configuration (i.e., position) dependent static equilibrium. This principle has been employed by many researchers (e.g., Whitney, 1972, Paul, 1972 and Thomas, 1981) to deal with loads applied to the end-effector of a serial manipulator. It can also be used to handle loads generated by such system components as springs and dampers as demonstrated by Benedict and Tesar (1978b) and Freeman (1980).

D'Alembert's principle is used in conjunction with the principle of virtual work to address the system's inertial dynamics. This approach to determining the generalized inertial loads is not unique to the author and has been

observed by several investigators (e.g., Kane, 1961 and Silver, 1982). One form of the equations resulting from this approach has even been referred to recently (by Kane and Levinson, 1983) as Kane's dynamic equations. The concept (alluded to as early as 1923 by Wittenbauer), however, is a natural consequence of the two principles and is usually referred to as the generalized principle of d'Alembert (see Lanczos, 1962, Meirovitch, 1970 and Lee, 1982).

Finally, the dynamic equations are expressed in terms of configuration (i.e., position) dependent coefficients operated on by the higher-order generalized input coordinate kinematics (i.e., velocity and acceleration). This result for the dynamic model is greatly facilitated by the use of kinematic influence coefficients and yields a highly geometric description of the system dynamics where the effect of the action of the generalized coordinates on each other is transparent. It should be noted that this transparency is of extreme importance (if not absolutely necessary) when dealing with the transfer of the dynamic model from one set of generalized coordinates to another.

The Principle of Virtual Work

This principle is the first variational principle in mechanics and aids in the transition from Newtonian (Vectorial) dynamics to Lagrangian (Analytical) dynamics. It is concerned with the static equilibrium of mechanical systems, and can be stated as: The work done by the applied

forces in infinitesimal reversible virtual displacements compatible with the system constraints is zero. To get a more analytical handle on this principle, consider a system containing N, P-dimensional, dependent motion parameters

$$ {}^{i}\underline{u} = ({}^{i}u^{1},\ {}^{i}u^{2},\ \ldots,\ {}^{i}u^{P})^{T}\ ,\ i = 1,2,\ldots,N \tag{2-119} $$

acted on by N, P-dimensional, applied loads

$$ {}^{i}\underline{T}^{u} = ({}^{i}T^{1},\ {}^{i}T^{2},\ \ldots,\ {}^{i}T^{P})^{T}\ ,\ i=1,2,\ldots,N \tag{2-120} $$

where the preceding superscript indicates which parameter is being considered, and the applied loads can be viewed as implicitly dependent on the generalized coordinates. Further, suppose that the system has M degrees-of-freedom and, hence, its motion can be completely described in terms of M generalized inputs,

$$ q = (q_{1},\ q_{2},\ldots,\ q_{M})^{T} \tag{2-121} $$

and that these coordinates are acted on by M generalized forces

$$ \underline{T}_{q} = (T_{1},\ T_{2},\ \ldots,\ T_{M})^{T} \tag{2-122} $$

For the system to be in static equilibrium, the principle of virtual work states that the virtual work (δW) must be zero, or

$$ \begin{aligned} \delta W &= \underline{T}_{q} \cdot \delta q + \sum_{i=1}^{N} {}^{i}\underline{T}^{u} \cdot \delta {}^{i}\underline{u} \\ &= (\delta q)^{T}\ \underline{T}_{q} + \sum_{i=1}^{N} (\delta {}^{i}\underline{u})^{T}\ {}^{i}\underline{T}^{u} \\ &= (\delta q)^{T}\ \underline{T}_{q} + \sum_{i=1}^{N} ((\delta q)^{T}[{}^{i}G^{u}_{q}]^{T})\,{}^{i}\underline{T}^{u} \\ &= (\delta q)^{T}\{\underline{T}_{q} + \sum_{i=1}^{N} [{}^{i}G^{u}_{q}]\ {}^{i}\underline{T}^{u}\} \\ &= 0 \end{aligned} \tag{2-123} $$

since, for the virtual displacements ($\delta{}^i\underline{u}$) to be compatible with the system constraints (see equations (2-7) and (2-8))

$$\delta{}^i\underline{u} = \sum_{n=1}^{M} \frac{\partial\, {}^i\underline{u}}{\partial q_n}\,\delta q_n = [{}^iG_q^u]\delta\underline{q} \qquad (2\text{-}124)$$

Now, because the virtual generalized displacements (δq^n) are all independent, hence arbitrary, equation (2-123) yields M independent equations for the required generalized loads as, in vector form

$$\underline{T}_q = -\sum_{i=1}^{N} [{}^iG_q^u]^T\, {}^i\underline{T}^u \equiv -\underline{T}_q^L \qquad (2\text{-}125)$$

where $\underline{T}_q^L$ is the effective load at the inputs due to all applied loads.

The Generalized Principle of D'Alembert

The statement of this principle is as follows: the total virtual work performed by the applied and inertial forces through infinitesimal virtual displacements, compatible with the system constraints, is zero. To see how this principle can be used to obtain the generalized input loads ($\underline{T}_q$) required to overcome the system's inertia and cause the desired motion, again consider the dependent parameters (${}^i\underline{u}$) of equation (2-119) where, now, they describe the kinematics of the system's mass parameters

$$[{}^iM^u] = P \times P\ ,\ i=1,2,\ldots,N \qquad (2\text{-}126)$$

This allows the system's momentum ($\underline{L}^u$) (both linear and angular) to be expressed by the N, P-dimensional, vectors (${}^i\underline{L}^u$) as

$$\underline{L}^u = \sum_{i=1}^{N} ({}^i\underline{L}^u) = \sum_{i=1}^{N} ([{}^iM^u]{}^i\dot{\underline{u}}) \qquad (2\text{-}127)$$

Further, assume that there exist N, P-dimensional, load vectors (${}^i\underline{T}^u$, i = 1,2,...,n) that if applied to the associated mass motion parameters (${}^i\underline{u}$) would cause the desired system kinematics (i.e., the position, velocity and acceleration of the ${}^i\underline{u}$). The generalized principle can now be written as

$$\sum_{i=1}^{N} ({}^i\underline{T}^u - {}^i\underline{L}^u) \cdot \delta{}^i\underline{u} = \sum_{i=1}^{N} ({}^i\underline{T}^u - \frac{d}{dt}([{}^iM^u]{}^i\dot{\underline{u}}))\cdot\delta{}^i\underline{u} = 0 \qquad (2\text{-}128)$$

yielding, for the virtual work (δW) done by these hypothetical applied loads (${}^i\underline{T}^u$), the following relationship:

$$\delta W = \sum_{i=1}^{N} {}^i\underline{T}^u \cdot \delta{}^i\underline{u} = \sum_{i=1}^{N} \frac{d}{dt}([{}^iM^u]{}^i\dot{\underline{u}}) \cdot \delta{}^i\underline{u} \qquad (2\text{-}129)$$

This result, in itself, is not very useful since the virtual displacements ($\delta{}^i\underline{u}$, i = 1,2,...,n) are not independent. However, when one recognizes that the virtual work can also be expressed in terms of a set of M generalized forces ($\underline{T}_q$) acting on a corresponding set of generalized coordinates ($\underline{q}$), which are independent, as (where here the generalized forces are, in essence, replacing the hypothetical applied loads instead of opposing them, as in equation (2-125) to obtain static equilibrium)

$$\delta W = \sum_{i=1}^{N} {}^i\underline{T}^u \cdot \delta{}^i\underline{u} = \underline{T}_q \cdot \delta q \qquad (2\text{-}130)$$

the following, very powerful, result occurs:

$$\underline{T}_q \cdot \delta\underline{q} = \sum_{i=1}^{N} \frac{d}{dt}([{}^iM^u]\,{}^i\underline{\dot{u}}) \cdot \delta\,{}^i\underline{u}$$
$$= \sum_{i=1}^{N} \frac{d}{dt}([{}^iM^u]\,{}^i\underline{\dot{u}}) \cdot [{}^iG^u_q]\delta\underline{q} \qquad (2\text{-}131)$$

or, following manipulations similar to equations (2-123), the required generalized forces are

$$\underline{T}_q = \sum_{i=1}^{N} [{}^iG^u_q]^T \{\frac{d}{dt}([{}^iM^u]\,{}^i\underline{\dot{u}})\} \qquad (2\text{-}132)$$

Investigation of equation (2-132) yields an interesting result. That is, the effect of the dynamics of each mass motion parameter (e.g., each link) on the generalized inputs can be considered independently (also see Silver, 1982, equation (25), where the dynamics of each link are expressed in terms of the Newton-Euler equations). In fact, the result of equation (2-132) can be obtained from the derivation of the generalized input loads required to offset a set of applied loads (see equations (2-123), (2-124) and (2-125)). To see this, simply replace the applied loads (${}^i\underline{T}^u$) in these equations with either the d'Alembert loads ($-\frac{d}{dt}([{}^iM^u]\,{}^i\underline{\dot{u}})$) or the negative of the Newton-Euler equations. In this light, one could view the result of equation (2-132) as being obtained from the virtual work of the d'Alembert loads.

Now, recalling equation (2-7) for (${}^i\underline{u}$), equation (2-132) can be written as

$$\underline{T}_q = \sum_{i=1}^{N} [{}^iG^u_q]^T \{\frac{d}{dt}([{}^iM^u][{}^iG^u_q]\underline{\dot{q}})\} \qquad (2\text{-}133)$$

Recognizing that the momentum vectors are functions of the generalized coordinate positions and velocities (i.e., $\underline{q}$ and $\underline{\dot{q}}$), one has that

$$\frac{d}{dt}([{}^{i}M^{u}][{}^{i}G^{u}_{q}]\underline{\dot{q}}) = \frac{\partial}{\partial \underline{q}}([{}^{i}M^{u}][{}^{i}G^{u}_{q}]\underline{\dot{q}})\underline{\dot{q}} + \frac{\partial}{\partial \underline{\dot{q}}}([{}^{i}M^{u}][{}^{i}G^{u}_{q}]\underline{\dot{q}})\underline{\ddot{q}} \tag{2-134}$$

The second of these terms follows immediately from equation (2-13) as

$$\frac{\partial}{\partial \underline{\dot{q}}}([{}^{i}M^{u}][{}^{i}G^{u}_{q}]\underline{\dot{q}})\underline{\ddot{q}} = [{}^{i}M^{u}][{}^{i}G^{u}_{q}]\underline{\ddot{q}} \tag{2-135}$$

If the mass parameters ($[{}^{i}M^{u}]$) are configuration independent (i.e., $[{}^{i}M^{u}] = f(q)$) (which is not the case when dealing with the effects of inertia in spatial devices, but including the generalization gives no insight here and the question will be dealt with specifically in the case of the serial manipulator), then recalling the development of equations (2-14) through (2-24), the first term of equation (2-134) becomes

$$\frac{\partial}{\partial \underline{q}}([{}^{i}M^{u}][{}^{i}G^{u}_{q}]\underline{\dot{q}})\underline{\dot{q}} = [{}^{i}M^{u}]\underline{\dot{q}}^{T}[{}^{i}H^{u}_{qq}]\underline{\dot{q}} \tag{2-136}$$

Substituting the results of equations (2-135) and (2-136) into equation (2-133) gives for the required input loads

$$\begin{aligned}\underline{T}q = &\sum_{i=1}^{N} \{[{}^{i}G^{u}_{q}]^{T}[{}^{i}M^{u}][{}^{i}G^{u}_{q}]\}\underline{\ddot{q}} \\ &+ \sum_{i=1}^{N} \{[{}^{i}G^{u}_{q}]^{T}[{}^{i}M^{u}]\underline{\dot{q}}^{T}[{}^{i}H^{u}_{qq}]\underline{\dot{q}}\}\end{aligned} \tag{2-137}$$

Note that for constant mass ($[{}^{i}M^{u}]$), this equation can be immediately obtained by substituting equation (2-27) for ($^{i}\underline{u}$) into equation (2-132). Now, equation (2-137) can be written in the general form for the generalized inertia loads ($\underline{T}^{I}_{q}$) as

$$\underline{T}^{I}_{q} = [I^{*}_{qq}]\ddot{\underline{q}} + \dot{\underline{q}}^{T}[P^{*}_{qqq}]\dot{\underline{q}} \tag{2-138}$$

where, the configuration dependent coefficient

$$[I^{*}_{qq}] = \sum_{i=1}^{N} [{}^{i}G^{u}_{q}]^{T}[{}^{i}M^{u}][{}^{i}G^{u}_{q}] = M \times M \tag{2-139}$$

is a positive definate matrix representing the effective inertia of the system as seen by the inputs. The system's kinetic energy is also expressable in terms of this coefficient (see Thomas, 1981) as

$$KE = \frac{1}{2}\,\dot{\underline{q}}^{T}[I^{*}_{qq}]\dot{\underline{q}} \tag{2-140}$$

The configuration dependent coefficient

$$[P^{*}_{qqq}] = \sum_{i=1}^{N} (([{}^{i}G^{u}_{q}]^{T}[{}^{i}M^{u}]) \bullet [{}^{i}H^{u}_{qq}]) = M \times M \times M^{\dagger} \tag{2-141}$$

is the inertia-power modeling matrix dealing with the effects of the velocity-related acceleration terms. Note that the subscripts on the model coefficients indicate, not only the dimension of the result, but also aid in investigating specific terms. For example, the n^{th} generalized inertia load is

$$T^{I}_{n} = [I^{*}_{nq}]\ddot{\underline{q}} + \dot{\underline{q}}^{T}[P^{*}_{nqq}]\dot{\underline{q}} \tag{2-142}$$

$$= [I^{*}_{qq}]_{n;}\ddot{\underline{q}} + \dot{\underline{q}}^{T}[P^{*}_{qqq}]_{n;;}\dot{\underline{q}}$$

where

$$[I^*_{nq}] = \sum_{i=1}^{N} [{}^iG^u_n]^T [{}^iM^u][{}^iG^u_q] \qquad (2\text{-}143)$$

and

$$[P^*_{nqq}] = \sum_{i=1}^{N} \{([{}^iG^u_n]^T [{}^iM^u]) \cdot [{}^iH^u_{qq}]\} \qquad (2\text{-}144)$$

It should be noted that the form given in equation (2-138) for the inertia related terms of the dynamic model is completely general and applies to all types of rigid-link mechanisms. The only differences lie in how the kinematic influence coefficients are obtained and, if the mass parameters ($[{}^iM^u]$) are configuration dependent, the coefficient $[P^*_{nqq}]$ contains an additional term (as will be seen in the serial manipulator development).

The Dynamic Equations

Finally, recalling equations (2-125) and (2-138), the generalized input forces (or torques) required to generate the desired system trajectory under load are given by

$$\begin{aligned} \underline{T}_q &= \underline{T}_q^I - T_q^L \\ &= [I^*_{qq}]\underline{\ddot{q}} + \underline{\dot{q}}^T [P^*_{qqq}]\underline{\dot{q}} - \sum_{i=1}^{N} [{}^iG^u_q]^T \, {}^i\underline{T}^u \end{aligned} \qquad (2\text{-}145)$$

This result shows the highly geometric nature of the dynamics of multi-body systems, and is precisely the reason why this representation for the dynamic model is chosen over the Newton-Euler form. While this form may, or may not, immediately yield the most efficient scheme for real-time

computation (Hollerbach, 1980), the insight that can be obtained from this geometric view is considered essential for real-time control. What is meant by this is that, if one does not understand the essence (which is geometric) of how the effects of the system parameters (e.g., system mass, end-effector loads) on the inputs vary, then it is unlikely that one can intelligently address the question of any type of control, much less real-time control. Regardless, as will be seen later, when dealing with cooperating manipulators (including walking machines and multi-fingered hands) the geometric form of equation (2-145) is extremely useful, if not altogether essential.

This form (equation (2-145)) is also convenient in addressing the question of dynamic simulation. Here, since the effective inertia matrix ($[I^*_{qq}]$) is positive definite (i.e., its inverse always exists),one can determine the system's response by solving equation (2-145) for the generalized accelerations as

$$\underline{\ddot{q}} = [I^*_{qq}]^{-1}(\underline{T}_q - \underline{\dot{q}}^T[P^*_{qqq}]\underline{\dot{q}} + \sum_{i=1}^{N}[{}^{i}G^u]^T\ {}^{i}\underline{T}^u) \qquad (2\text{-}146)$$

While any of a number of numerical integration routines can be employed to solve this equation, multi-step predictor-corrector methods (e.g., Adams) are suggested over single-step methods (e.g., Runge-Kutta) due to their greater efficiency regarding compute-time versus accuracy. This greater efficiency is desired because of the computational

complexity of the function evaluations required at each time step. It should be pointed out, however, that the effective inertia matrix is the only model coefficient necessary for the response prediction. The remaining terms are perhaps best obtained in terms of a recursive formulation due to the supposed greater computational efficiency afforded by that method (Walker and Orin, 1982).

Also, the influence coefficient model formulation allows one to directly address the question of design (Thomas, 1981). Here, the model parameters can be used in conjunction with classical optimization techniques to develop actuator sizing (Thomas et al., 1984) and stiffness (Thomas et al., 1985) criteria, as well as motion and load capacity due to actuator limitations (Thomas and Tesar, 1982a).

Dynamics of Serial Manipulators

The development of the dynamic equations presented here will follow the structure of the previous parts of this section. First, the effect of applied loads on the manipulator's generalized inputs ($\underline{\varphi}$) will be addressed using the principle of virtual work. Next, the systems inertial effects will be considered via the generalized principle of d'Alembert. Finally, the resulting dynamic equations are presented in the form of the generalized dynamic model of equation (2-145).

Applied loads

As with the kinematics, it is convenient to separate the rotational and translational properties when dealing with the dynamics of rigid bodies undergoing spatial motion. In this light, consider a set of M, 3-component, force vectors ($^j\underline{f}^P$, j = 1,2,...,M) and a set of M, 3-component, moment vectors ($\underline{m}^{jk}$, j = 1,2,...,M) applied to their respective translational ($^j\underline{P}$) and rotational ($\underline{\omega}^{jk}$) motion parameters. Now, immediately from the virtual work result of equation (2-125), the effect of the applied loads on the generalized input coordinates ($\underline{\varphi}$) is given by

$$\underline{T}^L_\varphi = \sum_{j=1}^{N} \{[^jG^P_\varphi]^T \; ^j\underline{f}^P + [G^{jk}_\varphi]^T \; \underline{m}^{jk}\} \tag{2-147}$$

where, the Jacobians are both 3xM matrices defined in equations (2-66) and (2-55).

Inertial effects

To address the effects of the system inertia one could, as previously implied, simply apply the principle of virtual work to the classical Newton-Euler equations of motion for a rigid body, yielding

$$\underline{T}^I_\varphi = \sum_{j=1}^{M} \{[^jG^c_\varphi]^T \, \bar{M}^{jk} \; ^j\underline{a}^c \tag{2-148}$$

$$+ [G^{jk}_\varphi]^T([II^{jk}] \, \underline{\alpha}^{jk} + \underline{\omega}^{jk} \times [II^{jk}]\underline{\omega}^{jk})\}$$

Here, $\bar{M}^{jk}$ and $[II^{jk}]$ are, respectively, the mass and global inertia matrix of link jk, with

$$ {}^{j}\underline{a}^{c} = [{}^{j}G^{c}_{\varphi}]\underline{\ddot{\varphi}} + \underline{\dot{\varphi}}^{T}[{}^{j}H^{c}_{\varphi\varphi}]\underline{\dot{\varphi}} \qquad (2\text{-}149) $$

being the acceleration of the center of mass of link jk ($^{j}\underline{c}$), and

$$ \underline{\omega}^{jk} = [G^{jk}_{\varphi}]\underline{\dot{\varphi}} \qquad (2\text{-}150) $$

and

$$ \underline{\alpha}^{jk} = [G^{jk}_{\varphi}]\underline{\ddot{\varphi}} + \underline{\dot{\varphi}}^{T}[H^{jk}_{\varphi\varphi}]\underline{\dot{\varphi}} \qquad (2\text{-}151) $$

are the absolute angular velocity and acceleration of link jk. Now, by substituting equations (2-149), (2-150) and (2-151) into equation (2-148) and algebraically manipulating the result one could obtain the desired general form of equation (2-138). While this operation gives the desired result it is felt that a slightly more rigorous approach, starting with the system momentum, gives more insight into the geometric nature of the solution.

The momentum of a rigid body (e.g., link jk) can be expressed in terms of two distinct vector quantities; these being the linear momentum ($\underline{p}^{jk}$) expressed in terms of the mass ($\bar{M}^{jk}$) and the velocity ($^{j}\underline{v}^{c}$) of the center of mass ($^{j}\underline{c}$), and the angular momentum ($\underline{L}^{jk}$) given in terms of the link's angular velocity ($\underline{\omega}^{jk}$) and the global inertia tensor ($[II^{jk}]$) about the centroid of the link.

The effective generalized inertia loads resulting from the rate of change of the system's linear momentum can be obtained directly from the development of equations (2-133)

through (2-141). This is possible since the link's mass ($\bar{M}^{jk}$) is constant with respect to the global reference (i.e., independent of the system's configuration) and can be expressed as

$$[{}^{j}M^{u}] = \bar{M}^{jk}[I]_{3x3} \qquad (2\text{-}152)$$

which allows one to write the link's linear momentum as

$$\underline{P}^{jk} = \bar{M}^{jk}[{}^{j}G^{c}_{\varphi}]\underline{\dot{\varphi}} \qquad (2\text{-}153)$$

The time rate of change is now

$$\underline{\dot{P}}^{jk} = M^{jk}([{}^{j}G^{c}_{\varphi}]\underline{\ddot{\varphi}} + \underline{\dot{\varphi}}^{T}[{}^{j}H^{c}_{\varphi\varphi}]\underline{\dot{\varphi}} \qquad (2\text{-}154)$$

and yields

$$^{P}\underline{T}^{I}_{\varphi} = \sum_{j=1}^{M} [{}^{j}G^{c}_{\varphi}]^{T}\, \underline{\dot{P}}^{jk} \qquad (2\text{-}155)$$

as the total effective load due to changing linear momentum. Expressing equation (2-155) in model form gives the load as

$$^{P}\underline{T}^{I}_{\varphi} = [{}^{P}I^{*}_{\varphi\varphi}]\underline{\ddot{\varphi}} + \underline{\dot{\varphi}}^{T}[{}^{P}\boldsymbol{P}^{*}_{\varphi\varphi\varphi}]\underline{\dot{\varphi}} \qquad (2\text{-}156)$$

with

$$[{}^{P}I^{*}_{\varphi\varphi}] = \sum_{j=1}^{M} [{}^{P}I^{jk}_{\varphi\varphi}] = \sum_{j=1}^{M} (\bar{M}^{jk}[{}^{j}G^{c}_{\varphi}]^{T}[{}^{j}G^{c}_{\varphi}]) = M \times M \qquad (2\text{-}157)$$

and

$$[{}^{P}P^{*}_{\varphi\varphi\varphi}] = \sum_{j=1}^{M} [{}^{P}P^{jk}_{\varphi\varphi\varphi}] = \sum_{j=1}^{M} \bar{M}^{jk}([{}^{j}G^{c}_{\varphi}]^{T} \bullet [{}^{j}H^{c}_{\varphi\varphi}]) = 3 \times M \times M \qquad (2\text{-}158)$$

where the Jacobians ($[{}^{j}G^{c}_{\varphi}]$) have shape (3xM) and the second-order coefficients ($[{}^{j}H^{c}_{\varphi\varphi}]$) have shape (3xMxM).

The development of the effective inertia loads (${}^{L}\underline{T}_{\varphi}$) caused by changes in the system's generalized angular momentum is more difficult. This is due to the fact that, while the local inertial properties ($[{}^{(j)}II^{jk}]$) of link jk are constant, the inertia properties ($[II^{jk}]$) expressed in terms of the global reference are not. As previously mentioned, this configuration dependence, resulting from the rate of change of the local reference frame in which the link's constant local inertia matrix is expressed, introduces an additional term into the inertial power matrix ($[{}^{L}P^{*}_{\varphi\varphi\varphi}]$).

With this in mind, it is perhaps best to initially view the global angular momentum of link jk ($\underline{L}^{jk}$) in terms of the link's angular momentum as expressed in its own body-fixed reference (${}^{(j)}\underline{L}^{jk}$). Recalling the use of the rotation matrix (T^{j}) illustrated by equations (2-47) and (2-58), one has

$$\underline{L}^{jk} = T^{j}\,{}^{(j)}\underline{L}^{jk} \tag{2-159}$$

Now, the local angular momentum is

$$ {}^{(j)}\underline{L}^{jk} = [{}^{(j)}II^{jk}]\,{}^{(j)}\underline{\omega}^{jk} \tag{2-160}$$

where, noting that the rotation matrix is orthogonal (i.e., $[T^{j}]^{T} = [T^{j}]^{-1}$), the link's locally expressed angular velocity is

$$ {}^{(j)}\underline{\omega}^{jk} = [T^{j}]^{T}\underline{\omega}^{jk} = [T^{j}]^{T}[G^{jk}_{\varphi}]\dot{\underline{\varphi}} \tag{2-161}$$

Combining equations (2-159), (2-160) and (2-161) gives the globally referenced angular momentum of link jk as

$$\underline{L}^{jk} = [T^j][{}^{(j)}II^{jk}][T^j]^T[G^{jk}_{\varphi}]\dot{\underline{\varphi}} \tag{2-162}$$

This is the desired form of the momentum since all configuration dependent terms are shown explicitly. Even so, notice that the global angular momentum of link jk ($\underline{L}^{jk}$) can be expressed in the same form as the local momentum (equation 2-160) as

$$\underline{L}^{jk} = [II^{jk}]\underline{\omega}^{jk} = [II^{jk}][G^{jk}_{\varphi}]\dot{\underline{\varphi}} \tag{2-163}$$

where, from equation (2-162)

$$[II^{jk}] = [T^j][{}^{(j)}II^{jk}][T^j]^T \tag{2-164}$$

is the globally referenced inertia dyadic.

As before, the momentum is seen to be a function of the generalized coordinate positions ($\underline{\varphi}$) and velocities ($\dot{\underline{\varphi}}$). Therefore, the time rate of change of link jk's angular momentum can be obtained from

$$\dot{\underline{L}}^{jk} = \frac{\partial \underline{L}^{jk}}{\partial \underline{\varphi}} \dot{\underline{\varphi}} + \frac{\partial \underline{L}^{jk}}{\partial \dot{\underline{\varphi}}} \ddot{\underline{\varphi}} \tag{2-165}$$

The second term in equation (2-165) is immediately seen to be

$$\frac{\partial \underline{L}^{jk}}{\partial \dot{\underline{\varphi}}} \ddot{\underline{\varphi}} = [II^{jk}][G^{jk}_{\varphi}] \ddot{\underline{\varphi}} \tag{2-166}$$

The position derivative is not as transparent and will be pursued in terms of the standard Jacobian formulation for vector differentials. Although there are three configuration dependent terms in the momentum expression, the differentiation will only be broken into two separate operations and addressed in the manner of Thomas (1981). This method gives, for the position derivative

$$\begin{aligned}
\frac{\partial \underline{L}^{jk}}{\partial \underline{\varphi}} &= \frac{\partial}{\partial \underline{\varphi}}([II^{jk}]\underline{\omega}^{jk}) \\
&= \frac{\partial}{\partial \underline{\varphi}}([II^{jk}][G_{\varphi}^{jk}]\underline{\dot{\varphi}}) \qquad (2\text{-}167) \\
&= \frac{\partial(T^{j})}{\partial \underline{\varphi}}\,[{}^{(j)}II^{jk}][T^{j}]^{T}[G^{jk}]\underline{\dot{\varphi}} \\
&\quad + [T^{j}]^{T}[{}^{(j)}II^{jk}]\frac{\partial}{\partial \underline{\varphi}}([T^{j}]^{T}[G_{\varphi}^{jk}]\underline{\dot{\varphi}})
\end{aligned}$$

To obtain the first partial in equation (2-167) it is beneficial to remember what the elements are that make up the columns of the rotation matrix:

$$T^{j} = [{}^{*}\underline{a}^{jk} \;\; {}^{*}\underline{s}^{j} \;\; x{}^{*}\underline{a}^{jk} \;\; {}^{*}\underline{s}^{j}] \qquad (2\text{-}168)$$

Here, one sees that the vectors whose direction cosines make up the columns of (T^{j}) are free vectors fixed relative to (i.e., in) link jk. In this light, recalling the more geometric formulation of the kinematic influence coefficients, the differentiation yields the familiar vector cross product result

$$\frac{\partial(T^j)}{\partial\varphi_n} = \underline{g}_n^{jk} \times T^j \ , \ n=1,2,\ldots,M \qquad (2\text{-}169)$$

Defining the remainder of the first term of equation (2-167), for convenience, as

$$\underline{b} \equiv [{}^{(j)}II^{jk}][T^j]^T[G_\varphi^{jk}]\dot{\underline{\varphi}} \qquad (2\text{-}170)$$

and observing the Jacobian convention, gives the n^{th} column of the first partial in equation (2-167) as

$$\left[\frac{\partial([T^j]\bar{b})}{\partial\underline{\varphi}}\right]_{;n} = \frac{\partial([T^j]\bar{b})}{\partial\varphi_n} = \underline{g}_n^{jk} \times [T^j]\underline{b} \qquad (2\text{-}171)$$

where, the vector ($\underline{b}$) is considered constant in this operation as indicated by the bar overstrike ($\underline{b}$). Introducing a generalized cross product notation (x) one has

$$[G_\varphi^{jk}]^T \times T^j\underline{b} \equiv [g_1^{jk} \times T^j\underline{b} \mid \ldots \mid g_M^{jk} \times T^j\underline{b}] \qquad (2\text{-}172)$$

where the first dimension (i.e., row) of the left argument operates across the first dimension of the right argument (i.e., across all rows). Now, recalling equations (2-164) and (2-170), equation (2-172) gives, for the first partial in equation (2-167), the relationship

$$\frac{\partial(T^j)}{\partial\underline{\varphi}}[{}^{(j)}II^{jk}][T^j]^T[G_\varphi^{jk}]\dot{\underline{\varphi}} = [G_\varphi^{jk}]^T \times [II^{jk}][G_\varphi^{jk}]\dot{\underline{\varphi}} \qquad (2\text{-}173)$$

Turning our attention to the second partial in equation (2-167), the first step (Thomas, 1981) is to recognize that since the rotation matrix (T^j) is orthogonal, i.e.,

$$T^j[T^j]^T = [I]_{3x3} = \text{constant}, \tag{2-174}$$

one has that

$$\frac{\partial}{\partial \underline{\varphi}}(T^j[T^j]^T[G^{jk}_{\varphi}]\underline{\dot{\varphi}}) = \frac{\partial}{\partial \underline{\varphi}}([G^{jk}_{\varphi}]\underline{\dot{\varphi}}) \tag{2-175}$$

Using the chain rule on the left-hand side of equation (2-175), and referring to equations (2-20) and (2-173), yields

$$[G^{jk}_{\varphi}]^T \times [G^{jk}_{\varphi}]\underline{\dot{\varphi}} + T^j \frac{\partial}{\partial \underline{\varphi}}([T^j]^T[G^{jk}_{\varphi}]\underline{\dot{\varphi}}) = \underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T \tag{2-176}$$

Rearranging and premultiplying by $[T^j]^T$ gives

$$\frac{\partial}{\partial \underline{\varphi}}([T^j]^T[G^{jk}_{\varphi}]\underline{\dot{\varphi}}) = [T^j]^T(\underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T - [G^{jk}_{\varphi}]^T \times [G^{jk}_{\varphi}]\underline{\dot{\varphi}}) \tag{2-177}$$

Direct investigation of the right-hand side of equation (2-177) shows that

$$\underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T - [G^{jk}_{\varphi}]^T \times [G^{jk}_{\varphi}]\underline{\dot{\varphi}} = \underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}] \tag{2-178}$$

Confirmation of the above result is left to the reader; however there is another way to show the validity of the final result which also points out an interesting result. Considering that the partial derivative of equation (2-177) is to be postmultiplied by $\underline{\dot{\varphi}}$, the ultimate usage of the relationship given by equation (2-178) can be justified by showing that

$$([G^{jk}_{\varphi}]^T \times [G^{jk}_{\varphi}]\underline{\dot{\varphi}})\underline{\dot{\varphi}} = \underline{0} \tag{2-179}$$

since

$$\underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T\underline{\dot{\varphi}} = \underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]\underline{\dot{\varphi}} \tag{2-180}$$

Recalling equations (2-55) and (2-172), and manipulating the result, one has the following:

$$\begin{aligned}([G^{jk}_{\varphi}]^T \times [G^{jk}_{\varphi}]\underline{\dot{\varphi}})\underline{\dot{\varphi}} &= ([G^{jk}_{\varphi}] \times \underline{\omega}^{jk})\underline{\dot{\varphi}} \\ &= [\underline{g}^{jk}_1 \times \underline{\omega}^{jk} \quad \ldots \quad \underline{g}^{jk}_M \times \underline{\omega}^{jk}]\underline{\dot{\varphi}} \\ &= \sum_{n=1}^{M}(\underline{g}^{jk}_n\dot{\varphi}_n) \times \underline{\omega}^{jk} \\ &= \underline{\omega}^{jk} \times \underline{\omega}^{jk} \\ &= ([G^{jk}_{\varphi}]\underline{\dot{\varphi}}) \times ([G^{jk}_{\varphi}]\underline{\dot{\varphi}}) \\ &= \underline{0}\end{aligned} \tag{2-181}$$

Therefore, while not rigorously verifying equation (2-178), the preceding proves that

$$\frac{\partial}{\partial\underline{\varphi}}([T^j]^T[G^{jk}_{\varphi}]\underline{\dot{\varphi}})\underline{\dot{\varphi}} = [T^j]^T(\underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]\underline{\dot{\varphi}}) \tag{2-182}$$

which is the term ultimately needed. Finally, by substituting equations (2-166), (2-173) and (2-182) into equation (2-165) and, recalling equation (2-164), the time rate of change of the angular momentum of link jk can be expressed as

$$\begin{aligned}\underline{\dot{L}}^{jk} &= [II^{jk}][G^{jk}]\underline{\ddot{\varphi}} + [II^{jk}_{\varphi\varphi}]\underline{\dot{\varphi}}^T[H^{jk}]\underline{\dot{\varphi}} \\ &\quad + ([G^{jk}_{\varphi}]^T \times [II^{jk}][G^{jk}_{\varphi}]\underline{\dot{\varphi}})\underline{\dot{\varphi}}\end{aligned} \tag{2-183}$$

Note that, by combining the first two terms of equation (2-183) and following the algebra of equation (2-181), equation (2-183) is an alternate form of Euler's equation of motion for a rigid body as expressed in the global reference

$$\underline{N} = \dot{\underline{L}}^{jk} = [II^{jk}]\underline{\alpha}^{jk} + \underline{\omega}^{jk} \times [II^{jk}]\underline{\omega}^{jk} \tag{2-184}$$

Now, referring to either equation (2-132) or (2-148), to obtain the generalized load required for this change in momentum (${}^{L}\underline{T}_{\varphi}^{I}$) one premultiplies equation (2-183) by the transpose of the link's first order rotational influence coefficient ($[G_{\varphi}^{jk}]^T$), yielding

$$\begin{aligned} {}^{L}\underline{T}_{\varphi}^{I} = \sum_{j=1}^{M} [G_{\varphi}^{jk}]^T \underline{L}^{jk} = \sum_{j=1}^{M} \{ & [G_{\varphi}^{jk}]^T [II^{jk}][G_{\varphi}^{jk}]\underline{\varphi} \\ & + [G_{\varphi}^{jk}]^T [II^{jk}]\underline{\varphi}^T [H_{\varphi\varphi}^{jk}]\underline{\varphi} \\ & + [G_{\varphi}^{jk}]^T ([G_{\varphi}^{jk}]^T \times [G_{\varphi}^{jk}]\underline{\varphi})\underline{\varphi}\} \end{aligned} \tag{2-185}$$

To get this equation into the desired form of equation (2-138) requires some manipulation of the velocity related terms. The first of these terms is readily accommodated by the generalized dot product (see Appendix A) giving

$$[G_{\varphi}^{jk}]^T [II^{jk}]\underline{\varphi}^T [H_{\varphi\varphi}^{jk}]\underline{\varphi} = \underline{\varphi}^T ([G_{\varphi}^{jk}]^T [II^{jk}] \bullet [H_{\varphi\varphi}^{jk}])\underline{\varphi} \tag{2-186}$$

The remaining term is a bit more obscure and is attacked with the aid of the following vector relationship:

$$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{c} \cdot \underline{a} \times \underline{b} \tag{2-187}$$

Recalling

$$\underline{L}^{jk} = [II^{jk}][G_{\varphi}^{jk}]\dot{\underline{\varphi}} \tag{2-188}$$

one has that

$$[G_{\varphi}^{jk}]^T([G_{\varphi}^{jk}]^T \times [II^{jk}][G_{\varphi}^{jk}]\dot{\underline{\varphi}})$$

$$= [G_{\varphi}^{jk}]^T([G_{\varphi}^{jk}]^T \times \underline{L}^{jk}) \tag{2-189}$$

$$= [G_{\varphi}^{jk}]^T([\underline{g}_1^{jk} \times \underline{L}^{jk} \mid \ldots \mid \underline{g}_M^{jk} \times \underline{L}^{jk}])$$

$$= \begin{bmatrix} \underline{g}_1^{jk} \cdot \underline{g}_1^{jk} \times \underline{L}^{jk} & \ldots & \underline{g}_1^{jk} \cdot \underline{g}_M^{jk} \times \underline{L}^{jk} \\ \\ \underline{g}_M^{jk} \cdot \underline{g}_1^{jk} \times \underline{L}^{jk} & \ldots & \underline{g}_M^{jk} \cdot \underline{g}_M^{jk} \times \underline{L}^{jk} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{L}^{jk} \cdot \underline{g}_1^{jk} \times \underline{g}_1^{jk} & \ldots & \underline{L}^{jk} \cdot \underline{g}_1^{jk} \times \underline{g}_M^{jk} \\ \\ \underline{L}^{jk} \cdot \underline{g}_M^{jk} \times \underline{g}_1^{jk} & & \underline{L}^{jk} \cdot \underline{g}_M^{jk} \times \underline{g}_M^{jk} \end{bmatrix}$$

$$= (\underline{L}^{jk})^T[[G_{\varphi}^{jk}]^T \times [G_{\varphi}^{jk}]]$$

where

$$[[G_{\varphi}^{jk}]^T \times [G_{\varphi}^{jk}]]_{n;;} = \underline{g}_n^{jk} \times [G_{\varphi}^{jk}] = 3 \times M \tag{2-190}$$

so

$$[G_{\varphi}^{jk}]^T([G_{\varphi}^{jk}]^T \times [II^{jk}][G_{\varphi}^{jk}]\dot{\underline{\varphi}})\dot{\underline{\varphi}} \tag{2-191}$$

$$= \dot{\underline{\varphi}}^T([G_{\varphi}^{jk}]^T[II^{jk}]([G_{\varphi}^{jk}]^T \times [G_{\varphi}^{jk}]))\dot{\underline{\varphi}}$$

Now, from equations (2-185), (2-186) and (2-191), one has

$$\underline{{}^{L}T}^{I}_{\varphi\varphi} = [{}^{L}I^{*}_{\varphi\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^{T}[{}^{L}P^{*}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} \qquad (2\text{-}192)$$

where

$$[{}^{L}I^{*}_{\varphi\varphi}] = \sum_{j=1}^{M}[{}^{L}I^{jk}_{\varphi\varphi}] = \sum_{j=1}^{M}([G^{jk}_{\varphi}]^{T}[II^{jk}][G^{jk}_{\varphi}]) \qquad (2\text{-}193)$$

and

$$[{}^{L}P^{*}_{\varphi\varphi\varphi}] = \sum_{j=1}^{M}[{}^{L}P^{jk}_{\varphi\varphi\varphi}] = \sum_{j=1}^{M}\{[G^{jk}_{\varphi}]^{T}[II^{jk}] \bullet [H^{jk}_{\varphi\varphi}])$$
$$+ ([G^{jk}_{\varphi}]^{T}[II^{jk}]([G^{jk}_{\varphi}]^{T} \times [G^{jk}_{\varphi}]))\} \qquad (2\text{-}194)$$

Finally, combining the effects of both the linear and angular momentum gives

$$\underline{T}^{I}_{\varphi} = [I^{*}_{\varphi\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^{T}[P^{*}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} \qquad (2\text{-}195)$$

for the total generalized inertia load, where

$$[I^{*}_{\varphi\varphi}] = [{}^{P}I^{*}_{\varphi\varphi}] + [{}^{L}I^{*}_{\varphi\varphi}] \qquad (2\text{-}196)$$
$$= \sum_{j=1}^{M}\{\bar{M}^{jk}[{}^{j}G^{c}_{\varphi}]^{T}[{}^{j}G^{c}_{\varphi}] + [G^{jk}_{\varphi}]^{T}[II^{jk}][G^{jk}_{\varphi}]\}$$

and

$$[P^{*}_{\varphi\varphi\varphi}] = [{}^{P}P^{*}_{\varphi\varphi\varphi}] + [{}^{L}P^{*}_{\varphi\varphi\varphi}] \qquad (2\text{-}197)$$
$$= \sum_{j=1}^{M}\{\bar{M}^{jk}([{}^{j}G^{c}_{\varphi}]^{T} \bullet [{}^{j}H^{c}_{\varphi\varphi}])$$
$$+ ([G^{jk}_{\varphi}]^{T}[II^{jk}] \bullet [H^{jk}_{\varphi\varphi}])$$
$$+ ([G^{jk}]^{T}[II^{jk}]([G^{jk}]^{T} \times [G^{jk}]))\}$$

The dynamic equations

The dynamic equations for the general M-degree of freedom serial manipulator can now be written in the desired general form of equation (2-145), giving the required generalized input load as

$$\underline{T}_\varphi = [I^*_{\varphi\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[P^*_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} - \sum_{j=1}^{M}([{}^jG^P_\varphi]^T\, {}^j\underline{f}^P + [G^{jk}_\varphi]^T\underline{m}^{jk} \qquad (2\text{-}198)$$

with the I* and P* matrices as defined in equations (2-196) and (2-197). Solving this equation for the generalized coordinate accelerations gives the simulation form of the dynamic equation

$$\ddot{\underline{\varphi}} = [I^*_{\varphi\varphi}]^{-1}\{\underline{T}_\varphi - \dot{\underline{\varphi}}^T[P^*_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} + \underline{T}^L_\varphi\} \qquad (2\text{-}199)$$

used to predict the motion of the manipulator resulting from the load state on the system.

This completes the development of the dynamic model for the general M-degree of freedom serial manipulator. The two second-order equations (2-198) and (2-199), or some other representation of these relationships, are the usual stopping points in the study of the motion of rigid-link mechanisms. While one can determine the input loads ($\underline{T}_\varphi$) (force or torque) required to cause a specified motion ($\varphi(t)$), or the response ($\varphi(t)$) of the system to the specified load state ($\underline{T}_\varphi$), the question of control is not immediately addressable. The reason for this is that these equations assume the instantaneous availability of finite

torque. This availability is not generally the case since the actuators themselves are governed by their own dynamic equations. In an effort to overcome this deficiency the following section, cognizant of modern linear control theory, is included.

The Linearization of the Dynamic Model

Many researchers (e.g., Kahn and Roth, 1971, Dubowsky and DesForges, 1979, Horowitz and Tomizuka, 1980, Liegeois et al., 1980, Armada and No, 1981, LeBorgne et al., 1981, Golla et al., 1981, Freund, 1982, Guez, 1982, Vukobratovic and Stokic, 1982a, and Stoten, 1983) have investigated (and are continuing to investigate) various methods of linear and nonlinear, adaptive and nonadaptive, control. While it appears likely that nonlinear adaptive control schemes hold the greatest promise for the ultimate real-time control of highly nonlinear systems such as robotic manipulators, the development of such a scheme is not the purpose of this section. The purpose of this section is, instead, to use the well established technique of linearization about a nominal motion trajectory, to develop a generalized state space model. Once this model is established, while again not the purpose here, one can apply the tremendous body of knowledge of modern control theory to address the feasibility of such control methods. Here, of primary concern would be the question of the time variance of the

coefficients in the state space model. In other words, for a given task (or trajectory), do the coefficients vary slowly enough to allow the system to be treated as time invariant with respect to controller design? While this type of formulation is not original (see Vukobratovic and Kircanski, 1982b, Lee and Lee, 1984, and Whitehead, 1984), it does not appear to have been fully investigated. Therefore, for possible future investigation (for specific short-term solutions) and completeness, this section presents the development of the generalized controlling state space equations for the serial manipulator, including actuator dynamics.

The Linearized Equations of Motion

The first step in the development of the controlling state space equations is the linearization of the dynamic equations with respect to the prescribed state variables (ignoring, for the moment, the actuator dynamics). Here, adopting the standard velocity-referenced model, the deviations in the generalized coordinate positions ($\delta\underline{\varphi}$) and velocities ($\delta\dot{\underline{\varphi}}$) make up the (2Mx1) state vector ($\underline{x}$), or

$$\underline{x} = \begin{pmatrix} \delta\underline{\varphi} \\ - - - \\ \delta\dot{\underline{\varphi}} \end{pmatrix} = \begin{pmatrix} \underline{\varphi} \text{ actual} - \underline{\varphi} \text{ nominal} \\ \dot{\underline{\varphi}} \text{ actual} - \dot{\underline{\varphi}} \text{ nominal} \end{pmatrix}$$

Now, noting that the generalized load ($\underline{T}_{\varphi}$) is a function of the generalized positions ($\underline{\varphi}$), velocities ($\dot{\underline{\varphi}}$) and accelerations ($\ddot{\underline{\varphi}}$), one has

$$\underline{T}_x = \delta\underline{T}_\varphi = \left.\frac{\partial \underline{T}_\varphi}{\partial \ddot{\underline{\varphi}}}\right|_{\underline{\varphi}(t)\text{nom.}} \delta\ddot{\underline{\varphi}} + \left.\frac{\partial \underline{T}_\varphi}{\partial \dot{\underline{\varphi}}}\right|_{\underline{\varphi}(t)\text{nom.}} \delta\dot{\underline{\varphi}} + \left.\frac{\partial \underline{T}_\varphi}{\partial \underline{\varphi}}\right|_{\underline{\varphi}(t)\text{nom.}} \delta\underline{\varphi} \qquad (2\text{-}201)$$

as the general equation for the linearized dynamics (the question of independent external load disturbance is addressed later). The task at hand, then, is to derive expressions for the partial derivatives in equation (2-201), to as great a degree as possible, in terms of the same general algebraic operations and parametric modeling coefficients employed in the dynamic equations themselves (to minimize the additional computational burden).

Writing the generalized load in the form of equation (2-198) gives

$$\underline{T}_\varphi = [I^*_{\varphi\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[P^*_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} - \sum_{j=1}^{M}[{}^jG^u_\varphi]^T\, {}^j\underline{T}^u \qquad (2\text{-}202)$$

where $({}^j\underline{T}^u)$ is a general applied load vector associated with link jk consisting of both force and moment components (and is assumed to be constant in the following derivations). This form allows one to obtain the first two terms in equation (2-201) directly from previous operations. Recognizing the similarity in the form of equations (2-27) and (2-202) and, recalling the development of equations (2-28) through (2-36), one has that

$$\frac{\partial \underline{T}_\varphi}{\partial \ddot{\underline{\varphi}}}\delta\ddot{\underline{\varphi}} = [I^*_{\varphi\varphi}]\delta\ddot{\underline{\varphi}} \qquad (2\text{-}203)$$

and

$$\frac{\partial \underline{T}_\varphi}{\partial \dot{\underline{\varphi}}} \delta\dot{\underline{\varphi}} = \dot{\underline{\varphi}}^T([P^*_{\varphi\varphi\varphi}] + [P^*_{\varphi\varphi\varphi}]^T)\delta\dot{\underline{\varphi}} \tag{2-204}$$

Now, to address the third term of equation (2-201), particularly in light of the previously described generalized vector dot (·) and cross (x) product operations, it is convenient to express the dynamics in terms of the Newton-Euler form of equations (2-147), (2-155) and (2-185) as

$$\underline{T}_\varphi = \sum_{j=1}^{M}([G^{jk}_\varphi]^T \dot{\underline{L}}^{jk} + [{}^jG^c_\varphi]^T \dot{\underline{P}}^{jk} - [{}^jG^u_\varphi]^T \, {}^j\underline{T}^u) \tag{2-205}$$

Investigating the last term of equation (2-205), where the preceding superscript (j) is dropped, one has the standard Jacobian result

$$\frac{\partial([G^u_\varphi]^T \underline{T}^u)}{\partial \underline{\varphi}} = \frac{\partial(\underline{T}^L_\varphi)}{\partial \underline{\varphi}} = \begin{bmatrix} \frac{\partial T^L_1}{\partial \varphi_1} & \frac{\partial T^L_1}{\partial \varphi_2} & \cdots & \frac{\partial T^L_1}{\partial \varphi_M} \\ \frac{\partial T^L_2}{\partial \varphi_1} & & & \\ \vdots & & & \\ \frac{\partial T^L_M}{\partial \varphi_1} & & & \frac{\partial T^L_M}{\partial \varphi_M} \end{bmatrix} \tag{2-206}$$

This gives the component in the m^{th} row and n^{th} column as

$$\frac{\partial T^L_m}{\partial \varphi_n} = \frac{\partial((\underline{T}^u)^T \cdot (\underline{g}^u_m))}{\partial \varphi_n} = (\underline{T}^u)^T \cdot \underline{h}^u_{nm} \tag{2-207}$$

or, for the total MxM matrix result, reintroducing the superscript (j), one obtains the expression

$$\frac{\partial([{}^{j}G^{u}_{\varphi}]^{T}\ {}^{j}\underline{T}^{u})}{\partial\underline{\varphi}} = ({}^{j}\underline{T}^{u})^{T} \bullet [{}^{j}H^{u}_{\varphi\varphi}]^{T} \qquad (2\text{-}208)$$

The contribution of the linear momentum can be determined, using the chain rule, from

$$\frac{\partial([{}^{j}G^{c}_{\varphi}]^{T}\dot{\underline{P}}^{jk})}{\partial\underline{\varphi}} = \frac{\partial([{}^{j}G^{c}_{\varphi}]^{T}\bar{\dot{\underline{P}}}^{jk})}{\partial\underline{\varphi}} + [{}^{j}G^{c}_{\varphi}]^{T}\ \frac{\partial(\dot{\underline{P}}^{jk})}{\partial\underline{\varphi}} \qquad (2\text{-}209)$$

The first term, where the change in momentum is considered constant (signified by the bar notation), follows directly from the previous derivation and yields

$$\frac{\partial([{}^{j}G^{c}_{\varphi}]^{T}\bar{\dot{\underline{P}}}^{jk})}{\partial\underline{\varphi}} = (\dot{\underline{P}}^{jk})^{T} \bullet [{}^{j}H^{c}_{\varphi\varphi}] \qquad (2\text{-}210)$$

with the symmetry in the translational (h,H) function observed. Recalling equation (2-154), one has that

$$\frac{\partial(\dot{\underline{P}}^{jk})}{\partial\underline{\varphi}} = \bar{M}^{jk}\ \frac{\partial}{\partial\underline{\varphi}}([{}^{j}G^{c}_{\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^{T}[{}^{j}H^{c}_{\varphi\varphi}]\dot{\underline{\varphi}}) \qquad (2\text{-}211)$$

which, referring to equation (2-38) and (2-41), becomes

$$\frac{\partial(\underline{P}^{jk})}{\partial\underline{\varphi}} = \bar{M}^{jk}(\ddot{\underline{\varphi}}^{T}[{}^{j}H^{c}_{\varphi\varphi}] + \dot{\underline{\varphi}}^{T}[{}^{j}D^{c}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}}) \qquad (2\text{-}212)$$

Now, premultipling equation (2-212) by the Jacobian transpose and substituting, with equation (2-210), the result into equation (2-209) gives

$$\frac{\partial([{}^{j}G^{c}_{\varphi}]^{T}P^{jk})}{\partial\underline{\varphi}} = (\dot{\underline{P}}^{jk})^{T} \bullet [{}^{j}H^{c}_{\varphi\varphi}] \qquad (2\text{-}213)$$

$$+ \bar{M}^{jk}[{}^jG^c_\varphi]^T(\ddot{\underline{\varphi}}^T[{}^jH^c_{\varphi\varphi}])$$

$$+ \bar{M}^{jk}[{}^jG^c_\varphi]^T(\dot{\underline{\varphi}}^T[{}^jD^c_{\varphi\varphi\varphi}]\dot{\underline{\varphi}})$$

Application of the generalized inner product to the second term of this equation yields

$$\bar{M}^{jk}[{}^jG^c_\varphi]^T(\ddot{\underline{\varphi}}^T[{}^jH^c_{\varphi\varphi}]) = \ddot{\underline{\varphi}}^T(\bar{M}^{jk}[{}^jG^c_\varphi]^T \bullet [{}^jH^c_{\varphi\varphi}]) \tag{2-214}$$

which, recalling equation (2-158), gives the fortuitous result

$$\bar{M}^{jk}[{}^jG^c_\varphi]^T(\ddot{\underline{\varphi}}^T[{}^jH^c_{\varphi\varphi}]) = \ddot{\underline{\varphi}}^T[{}^Pp^{jk}_{\varphi\varphi\varphi}] \tag{2-215}$$

Finally, substituting equation (2-215) into (2-213) one obtains the expression

$$\begin{aligned} \frac{\partial([{}^jG^c_\varphi]^T\dot{\underline{P}}^{jk})}{\partial\underline{\varphi}} &= (\dot{\underline{P}}^{jk})^T \bullet [{}^jH^c_{\varphi\varphi}] \\ &+ \ddot{\underline{\varphi}}^T[{}^Pp^{jk}_{\varphi\varphi\varphi}] \\ &+ \bar{M}^{jk}[{}^jG^c_\varphi]^T(\dot{\underline{\varphi}}^T[{}^jD^c_{\varphi\varphi\varphi}]\dot{\underline{\varphi}}) \end{aligned} \tag{2-216}$$

Now, recalling the result of equation (2-208) and the form of equation (2-209), the contribution of a link's angular momentum to the last term in equation (2-201) can be written immediately as

$$\frac{\partial([G^{jk}_\varphi]^T\dot{\underline{L}}^{jk})}{\partial\underline{\varphi}} = (\dot{\underline{L}}^{jk})^T \bullet [H^{jk}_{\varphi\varphi}]^T + [G^{jk}_\varphi]^T \frac{\partial(\dot{\underline{L}}^{jk})}{\partial\underline{\varphi}} \tag{2-217}$$

To obtain the final partial derivative it is perhaps most convenient, especially considering previous derivations, to use Euler's form for the change in angular momentum (see equation(2-184)), or

$$\frac{\partial(\dot{\underline{L}}^{jk})}{\partial\underline{\varphi}} = \frac{\partial}{\partial\underline{\varphi}}([II^{jk}]\underline{\alpha}^{jk} + \underline{\omega}^{jk} \times [II^{jk}]\underline{\omega}^{jk}) \quad (2\text{-}218)$$

Addressing the first term one has (see the development presented in equations (2-167) through (2-177))

$$\frac{\partial}{\partial\underline{\varphi}}([II^{jk}]\underline{\alpha}^{jk}) = [G^{jk}_{\varphi}]^T \times [II^{jk}]\underline{\alpha}^{jk} + [II^{jk}]\left(\frac{\partial\underline{\alpha}^{jk}}{\partial\underline{\varphi}} - [G^{jk}_{\varphi}]^T \times \underline{\alpha}^{jk}\right) \quad (2\text{-}219)$$

Recalling equation (2-70) for link jk's angular acceleration and the form of equation (2-43) for the first geometric derivative of the acceleration allows one to rewrite equation (2-219) as

$$\begin{aligned}\frac{\partial}{\partial\underline{\varphi}}([II^{jk}]\underline{\alpha}^{jk}) = {} & ([G^{jk}_{\varphi}]^T \times [II^{jk}]([G^{jk}_{\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]\dot{\underline{\varphi}})) \\ & + [II^{jk}]\{(\ddot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T + \dot{\underline{\varphi}}^T[D^{jk}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} \\ & - [G^{jk}_{\varphi}]^T \times ([G^{jk}_{\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]\dot{\underline{\varphi}})\}\end{aligned} \quad (2\text{-}220)$$

Utilizing the result of equation (2-178), equation (2-220) can be reduced slightly to

$$\begin{aligned}\frac{\partial}{\partial\underline{\varphi}}([II^{jk}]\underline{\alpha}^{jk}) = {} & ([G^{jk}_{\varphi}]^T \times [II^{jk}]([G^{jk}_{\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]\dot{\underline{\varphi}})) \\ & + [II^{jk}](\ddot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}] + \dot{\underline{\varphi}}^T[D^{jk}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} - [G^{jk}_{\varphi}]^T \times \dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]\dot{\underline{\varphi}})\end{aligned} \quad (2\text{-}221)$$

Now, returning to equation (2-218), one has that

$$\frac{\partial}{\partial \underline{\varphi}}(\underline{\omega}^{jk} \times [II^{jk}]\underline{\omega}^{jk}) = \frac{\partial}{\partial \underline{\varphi}}(\underline{\omega}^{jk} \times \underline{L}^{jk}) \tag{2-222}$$

$$= \frac{\partial \underline{\omega}^{jk}}{\partial \underline{\varphi}} \times \underline{L}^{jk} + \underline{\omega}^{jk} \times \frac{\partial \underline{L}^{jk}}{\partial \underline{\varphi}}$$

Substituting the results of equations (2-20) and (2-167), respectively, for the partial derivatives in equation (2-222) give

$$\frac{\partial}{\partial \underline{\varphi}}(\underline{\omega}^{jk} \times \underline{L}^{jk}) = (\dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T) \times \underline{L}^{jk}$$
$$+ \underline{\omega}^{jk} \times ([G^{jk}_{\varphi}]^T \times \underline{L}^{jk} + [II^{jk}](\dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}])) \tag{2-223}$$

Combining the results of equations (2-221) and (2-223) to obtain an expression for equation (2-218) and substituting this expression into equation (2-217) yields, in light of equation (2-194) and the generalized vector dot (•) and cross (x) products, the 3xM matrix

$$\frac{\partial}{\partial \underline{\varphi}}([G^{jk}_{\varphi}]^T\dot{\underline{L}}^{jk}) = (\dot{\underline{L}}^{jk})^T \bullet [H^{jk}_{\varphi\varphi}]^T$$
$$+ \ddot{\underline{\varphi}}^T[L^{pjk}_{\varphi\varphi\varphi}] \tag{2-224}$$
$$+ [G^{jk}_{\varphi}]^T\{([G^{jk}_{\varphi}]^T \times \dot{\underline{\varphi}}^T([II^{jk}] \bullet [H^{jk}_{\varphi\varphi}])\dot{\underline{\varphi}})$$
$$+ ((\dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T \times [II^{jk}][G^{jk}_{\varphi}]\dot{\underline{\varphi}})$$
$$+ (([G^{jk}_{\varphi}]\dot{\underline{\varphi}})^T \times ([II^{jk}](\dot{\underline{\varphi}}^T[H^{jk}_{\varphi\varphi}]^T)$$
$$+ [G^{jk}_{\varphi}]^T \times ([II^{jk}][G^{jk}_{\varphi}]\dot{\underline{\varphi}})))$$

$$+ [II^{jk}]((\underline{\dot{\varphi}}^T[D^{jk}_{\varphi\varphi\varphi}]\underline{\dot{\varphi}} - [G^{jk}_{\varphi}]^T \times \underline{\dot{\varphi}}^T[H^{jk}_{\varphi\varphi}]\underline{\dot{\varphi}})\}$$

Finally, substituting the results of equations (2-203), (2-204), (2-208), (2-216) and (2-224) into equation (2-201) one has the general linearized dynamics given by (where all coefficients are evaluated at the nominal generalized coordinate values)

$$\delta\underline{T}_{\varphi} = \left[\frac{\partial \underline{T}_{\varphi}}{\partial \underline{\ddot{\varphi}}}\right]\delta\underline{\ddot{\varphi}} + \left[\frac{\partial \underline{T}_{\varphi}}{\partial \underline{\dot{\varphi}}}\right]\delta\underline{\dot{\varphi}} + \left[\frac{\partial \underline{T}_{\varphi}}{\partial \underline{\varphi}}\right]\delta\underline{\varphi} \tag{2-225}$$

$$= [I^*_{\varphi\varphi}]\delta\underline{\ddot{\varphi}} + [\underline{\dot{\varphi}}^T([P^*_{\varphi\varphi\varphi}] + [P^*_{\varphi\varphi\varphi}]^T)]\delta\underline{\dot{\varphi}}$$

$$+ [\sum_{j=1}^{M}\{-({}^j\underline{T}^u)^T \bullet [{}^jH^u_{\varphi\varphi}]^T + (\underline{\dot{P}}^{jk})^T \bullet [{}^jH^c_{\varphi\varphi}]^T$$

$$+ (\underline{\dot{L}}^{jk})^T \bullet [H^{jk}_{\varphi\varphi}]^T + [{}^jG^c_{\varphi}]^T \frac{\partial}{\partial \underline{\varphi}}(\underline{\dot{P}}^{jk})$$

$$+ [G^{jk}_{\varphi}]^T \frac{\partial}{\partial \underline{\varphi}}(\underline{\dot{L}}^{jk})\}]\ \delta\underline{\varphi}$$

Notice that, if the load (${}^j\underline{T}^u$) is a known function of the generalized coordinates, the effect can be easily accounted for. For example, if the load is a function of velocity the second term in equation (2-201) will become

$$\frac{\partial T_{\varphi}}{\partial \underline{\dot{\varphi}}}\ \delta\underline{\dot{\varphi}} = [\underline{\dot{\varphi}}^T([P^*_{\varphi\varphi\varphi}] + [P^*_{\varphi\varphi\varphi}]^T) - \sum_{j=1}^{M}[{}^jG^u_{\varphi}]^T \frac{\partial\, {}^j T^u}{\partial \underline{\dot{\varphi}}}]\delta\underline{\dot{\varphi}} \tag{2-226}$$

Also, if the load itself varies from its nominal value, which is likely, equation (2-201) contains the additional term

$$\sum_{j=1}^{M} \frac{\partial T_{\varphi}}{\partial_j T_u} = -\sum_{j=1}^{M} [{}^jG^u_{\varphi}]^T \delta_j \underline{T}_u \tag{2-227}$$

Investigation of equation (2-225) shows that no new computations are required for the first two coefficient matrices however; for the third (i.e., $[\partial \underline{T}_{\varphi}/\partial \underline{\varphi}]$) there appears to be a considerable additional requirement. In view of this, especially with recognition of the considerable similarity in the basic nature of the various components making up this coefficient, a detailed investigation of this matrix to determine the minimal reduced equation is warranted. However, since the linearized model coefficients are based on the nominal trajectory and determined off-line, this effort is left for future work.

Actuator Dynamics

In this section only one of many types (e.g., electro-mechanical, hydraulic, pneumatic, etc.) of actuators is investigated. For additional information on actuators and their dynamics the reader is referred to Borovac et al. (1983), Electrocraft (1978) and other sources.

The specific type of actuator addressed here is the separately excited d-c motor (Myklebust, 1982). In this device, the field current is held constant and speed control is accomplished by control of the armature terminal voltage. The controlling equations for this type of device are

$$L\frac{d\underline{i}}{dt} + Ri + C_E\dot{q} = V \qquad (2\text{-}228)$$

$$\bar{I}\ddot{q} + B\dot{q} - C_M i = -Tq$$

for a single actuator or, in matrix form, for all M actuators

$$[L_q]\dot{\underline{i}} + [Rq]\underline{i} + [C_{Eq}]\dot{\underline{q}} = \underline{V} \qquad (2\text{-}229)$$

$$[\bar{I}_q]\ddot{\underline{q}} + [B_q]\dot{\underline{q}} - [C_{Mq}]\underline{i} = -\underline{T}_q$$

where $\underline{i}$ is the Mx1 vector of armature currents (amp), $\underline{v}$ is the Mx1 vector of terminal voltage (volt), $[R_q]$ is the MxM diagonal matrix of armature resistances (Ohm), $[L_q]$ is the MxM diagonal matrix of armature inductances (Henry), q is the Mx1 vector of motor output shaft angles (rad), $[\bar{I}_q]$ is the constant MxM diagonal matrix of rotor inertias (in.-lb_f-sec^2), $[B_q]$ is the MxM diagonal matrix of rotor damping (in.-lb_f-sec), $[C_{Eq}]$ is the MxM diagonal matrix of motor speed constants (volt-sec), $[C_{Mq}]$ is the MxM diagonal matrix of motor torque constants (in.-lb_f/amp) and $\underline{T}_q$ is the Mx1 vector of load torques (in.-lb_f).

Addressing the actuator dynamics one must relate the M output shaft parameters (q) to the M generalized joint coordinates ($\underline{\varphi}$). Here, the following assumptions have been made: of constant transmission between the motor ($\underline{q}$) and joint ($\underline{\varphi}$) parameters (i.e., the Jacobian $[G_q^{\varphi}]$ is constant), zero compliance in the drive train, and zero backlash in the drive train. Recalling equations (2-7), (2-27) and (2-130) allows one to write

$$\underline{\dot{\varphi}} = [G_q^{\varphi}]\underline{\dot{q}}$$

$$\underline{\ddot{\varphi}} = [G_q^{\varphi}]\underline{\ddot{q}}\ , \ [G_q^{\varphi}] = \text{constant} \qquad (2\text{-}230)$$

and

$$\underline{T}_q = [G_q^{\varphi}]^T\underline{T}_{\varphi} \qquad (2\text{-}231)$$

where the MxM Jacobian ($[G_q^{\varphi}]$) is dependent on the specific drive trains employed. Here, the entry in row m, column n relates the motion of the m^{th} joint parameter (φ_m) to the n^{th} motor shaft (q_n). Typically, this matrix is diagonal since each joint is usually controlled by a single associated motor. In fact, for direct drive devices this Jacobian is simply the MxM identity matrix. At any rate, having obtained the Jacobian ($[G_q^{\varphi}]$), and recognizing that it is the joint motion (not the motor shaft) that one wishes to control, equation (2-230) is used to find the desired relations

$$\underline{\dot{q}} = [G_q^{\varphi}]^{-1}\underline{\dot{\varphi}} \qquad (2\text{-}232)$$

and

$$\underline{\ddot{q}} = [G_q^{\varphi}]^{-1}\underline{\ddot{\varphi}} \qquad (2\text{-}233)$$

Now, substituting equations (2-231), (2-232) and (2-233) into equation (2-229) yields

$$[L_q]\underline{\dot{i}} + [R_q]\underline{i} + [C_{Eq}][G_q^{\varphi}]^{-1}\underline{\dot{\varphi}} = \underline{V} \qquad (2\text{-}234)$$

$$[\bar{I}_q][G_q^{\varphi}]^{-1}\underline{\ddot{\varphi}} + [B_q][G_q^{\varphi}]^{-1}\underline{\dot{\varphi}} - [C_{Mq}]\underline{i} = -[G_q^{\varphi}]^T\underline{T}_{\varphi}$$

for the motor dynamics expressed in terms of the joint parameters. Finally, the actuator dynamics (equation (2-232)) can be expressed as

$$[L_q]\underline{\dot{i}} + [R_q]\underline{i} + [C_{E\varphi}]\underline{\dot{\varphi}} = \underline{V} \qquad (2\text{-}235)$$

and

$$-[\bar{I}_{\varphi}]\underline{\ddot{\varphi}} - [B_{\varphi}]\underline{\dot{\varphi}} + [C_{M\varphi}]\underline{i} = \underline{T}_{\varphi} \qquad (2\text{-}236)$$

where the second of equations (2-234) was premultiplied by minus the inverse transpose of the Jacobian and, for example, where

$$[\bar{I}_{\varphi}] \equiv [G_q^{\varphi}]^{-T}[\bar{I}_q][G_q^{\varphi}]^{-1} \qquad (2\text{-}237)$$

One should note that, with the stated assumptions, all the coefficient matrices on the left-hand side of equations (2-234) and (2-236) are independent of the generalized coordinate positions ($\underline{\varphi}$) and, since the variation of the motor parameters is not addressed, are considered constant.

Having obtained the desired expressions for the actuator dynamics, one can now determine the nominal input voltages ($\underline{V}$) required to drive the system along the specified nominal trajectory ($\underline{\varphi}(t)$). Recalling equation (2-202) and solving equation (2-236) for the current gives

$$\underline{i} = [C_{M\varphi}]^{-1}(\underline{T}_{\varphi} + [B_{\varphi}]\underline{\dot{\varphi}}) \qquad (2\text{-}238)$$

where the effective rotor inertia ($[\bar{I}_{\varphi}]$) has been included in the effective inertia matrix ($[I^{*}_{\varphi\varphi}]$). Now,

differentiating equation (2-238) with respect to time, and recalling equation (2-225), yields

$$\underline{i} = [C_{M\varphi}]^{-1}\{\frac{\partial \underline{T}_\varphi}{\partial \ddot{\underline{\varphi}}}\dddot{\underline{\varphi}} + \left(\frac{\partial \underline{T}_\varphi}{\partial \dot{\underline{\varphi}}} + [B_\varphi]\right)\ddot{\underline{\varphi}} + \frac{\partial \underline{T}_\varphi}{\partial \underline{\varphi}}\dot{\underline{\varphi}}\} \qquad (2\text{-}239)$$

Finally, assuming that the nominal joint kinematics are known up to the third order (i.e., $\underline{\varphi}(t)$, $\dot{\underline{\varphi}}(t)$, $\ddot{\underline{\varphi}}(t)$ and $\dddot{\underline{\varphi}}(t)$ are known), equations (2-238) and (2-239) can be substituted into equation (2-235) to obtain the nominal voltage ($\underline{V}$) as

$$\underline{V}_{nominal} = \underline{V}(\underline{\varphi},\dot{\underline{\varphi}},\ddot{\underline{\varphi}},\dddot{\underline{\varphi}})_{nominal} \qquad (2\text{-}240)$$

As with equation (2-225) for the linearized generalized load ($\delta\underline{T}_\varphi$), equation (2-240) could possible be reduced to a minimum generic form of the class of equation (2-145) ($\underline{T}_\varphi$). Again, this question will be left open for now since the immediate usage of the linearized scheme only required off-line computation of equation (2-240).

State Space Representation

Having incorporated the manipulator dynamics into the describing equations for the actuator dynamics, and having determined the nominal input (voltage) requirements one can proceed to develop the state space model of the system. In the case of the d-c motor driven manipulator, the state variables are chosen to be the deviations in the generalized coordinate positions ($\delta\underline{\varphi}$) and velocities ($\delta\dot{\underline{\varphi}}$) and the deviation in motor currents ($\delta\underline{i}$).

With this selection of the state variables the state space model is of the form

$$\begin{pmatrix} \delta\dot{\underline{\varphi}} \\ \delta\ddot{\underline{\varphi}} \\ \delta\dot{\underline{i}} \end{pmatrix} = \left[\begin{array}{c|c|c} F_{11} & F_{12} & F_{13} \\ \hline F_{21} & F_{22} & F_{23} \\ \hline F_{31} & F_{32} & F_{33} \end{array}\right] \begin{pmatrix} \delta\underline{\varphi} \\ \delta\dot{\underline{\varphi}} \\ \delta\underline{i} \end{pmatrix} + \left[\begin{array}{c} G_{11} \\ \hline G_{21} \\ \hline G_{31} \end{array}\right] \delta\underline{V} \qquad (2\text{-}241)$$

where each of the partition matrices (F_{ij}, G_{ij}) is MxM. These matrices are determined by considering the generalized velocities independent, and from equations (2-235) and (2-236). First

$$\dot{\underline{\varphi}} = \dot{\underline{\varphi}} \qquad (2\text{-}242)$$

gives

$$\delta\dot{\underline{\varphi}} = \delta\dot{\underline{\varphi}} \qquad (2\text{-}243)$$

so

$$[F_{11}] = [F_{13}] = [G_{11}] = [0] \qquad (2\text{-}244)$$

and

$$[F_{12}] = [I]_{MxM} \qquad (2\text{-}245)$$

Next, from equation (2-236)

$$[C_{M\varphi}]\delta\underline{i} = \delta\underline{T}_{\varphi} + [\bar{I}_{\varphi}]\delta\ddot{\underline{\varphi}} + [B_{\varphi}]\delta\dot{\underline{\varphi}} \qquad (2\text{-}246)$$

or, recalling equation (2-225)

$$[C_{M\varphi}]\delta\underline{i} = [I^{*}_{\varphi\varphi}]\delta\ddot{\underline{\varphi}} + \left(\frac{\partial\underline{T}_{\varphi}}{\partial\dot{\underline{\varphi}}} + [B_{\varphi}]\right)\delta\dot{\underline{\varphi}} + \left(\frac{\partial\underline{T}_{\varphi}}{\partial\underline{\varphi}}\right)\delta\underline{\varphi} \qquad (2\text{-}247)$$

which, when solved for $\delta\underline{\varphi}$, yields

$$[F_{21}] = -[I^{*}_{\varphi\varphi}]^{-1}\left(\frac{\partial T_{\varphi}}{\partial\underline{\varphi}}\right) \qquad (2\text{-}248)$$

$$[F_{22}] = -[I^{*}_{\varphi\varphi}]^{-1}\left(\frac{\partial\underline{T}_{\varphi}}{\partial\dot{\underline{\varphi}}} + [B_{\varphi}]\right)$$

$$[F_{23}] = [I^{*}_{\varphi\varphi}]^{-1}[C_{M\varphi}]$$

and

$$[G_{21}] = [0] \tag{2-249}$$

Finally, from equation (2-235)

$$[L_q]\delta\dot{\underline{i}} + [R_q]\delta\underline{i} + [C_{E\varphi}]\delta\dot{\underline{\varphi}} = \delta\underline{V} \tag{2-250}$$

which, when solved for $\delta\underline{i}$, gives

$$\begin{aligned} [F_{31}] &= [0] \\ [F_{32}] &= -[L_q]^{-1}[C_{E\varphi}] \\ [G_{33}] &= -[L_q]^{-1}[R_q] \end{aligned} \tag{2-251}$$

and

$$[G_{31}] = [L_q]^{-1} \tag{2-252}$$

where the inverse of the constant diagonal matrix $[L_q]$ is simply the diagonal matrix whose entries are inverted.

With the state space model describing the characteristics of the system about some nominal trajectory fully established, the questions associated with controller design can now be addressed. This is, again, not the purpose here and the reader is referred to the extensive work (Vukobratovic and Stokic, 1982a) and the work of Whitehead (1984) for discretization and potential controller development.

Note

[1]The dot (•) operator will prove to be an extremely powerful tool in the analyses presented throughout this work, and hence, will be dealt with in an appendix for easy reference. It is a generlized inner (or dot) product and is developed, along with the transition from equation (2-137) to equation (2-141), in Appendix A.

CHAPTER III

TRANSFER OF GENERALIZED COORDINATES

Consider the development of the general M-degree of freedom mechanism presented in Chapter II. Recalling, in particular, the sections dealing with the generalized kinematics and dynamics, one sees that the equations (S_q) describing the mechanics of such devices (e.g., equation (2-145) for the generalized load $\underline{T}_q$) can be expressed in terms of any desired set ($\underline{q} = (q_1, q_1, \ldots, q_M)^T$) of generalized coordinates once the kinematic influence coefficients (relating the dependent system parameters ($\underline{u}$) to that particular coordinate set ($\underline{q}$)) are known. Therefore, at least conceptually, the kinematics, dynamics and control of these systems can be addressed in generic form from any set ($\underline{q}$) of generalized coordinates that one desires, with the only difficulty being the determination of the required influence coefficients. Unfortunately, the determination of these required influence coefficients (directly) in terms of the desired coordinates ($\underline{q}$) may be (and often is) extremely difficult and impractical, if not altogether impossible. Fortunately however, most (if not all) conceivable mechanisms can be directly modeled (i.e., the influence coefficients can be directly obtained) from at least one of the many possible sets of generalized

coordinates. Now considering such a case, what one would like to do (see Benedict and Tesar,1971 and Freeman and Tesar, 1982) is, first obtain the system model (S_{φ}) with respect to some initial set of coordinates ($\underline{\varphi}$) for which the influence coefficients (including those relating the desired coordinates ($\underline{q}$) to the initial coordinates ($\underline{\varphi}$)) are easily determined and then, use this information (i.e., S_{φ}), along with the coefficients relating ($\underline{q}$) to ($\underline{\varphi}$), to determine the desired relationships (S_q).

To illustrate this situation, consider the general six-degree of freedom serial manipulator (i.e., M = 6) presented in Chapter II. Further, suppose that one desires to consider the system in terms of six generalized coordinates ($\underline{q}$) associated with the six end-effector freedoms since, after all, it is the hand that one typically wishes to control. For all but the simplest of systems, (e.g., partitionable systems such as those treated by Paul et al., 1981a and 1981b and Hollerbach and Sahar, 1983) where the inverse kinematics solution is readily available, the determination of the describing equations (S_q) directly in terms of the end-effector coordinates ($\underline{q}$) is, at best, extremely complicated (I know of no such solution existing in the literature). However, as shown in Chapter II, if one investigates the serial manipulator in terms of the relative joint angles ($\underline{\varphi}$), the describing equations (S_q) result from simple vector operations and are relatively easy to obtain. Also, since the end-effector coordinates ($\underline{q}$) can be viewed

as simply a particular set of dependent system parameters ($\underline{u}$), the kinematic influence coefficients relating the hand motion ($\underline{q}$) to the relative joint angles ($\underline{\varphi}$) are readily available. Finally, as will be shown later, the initial model information (i.e., S_φ), along with the known generalized coordinate relationships (i.e., $[G_\varphi]$ and higher order coefficients), can be used in a straight forward manner to determine the desired description (S_q) of the system interactions.

The development of this transfer (i.e., $S_\varphi \rightarrow S_q$) of generalized coordinates procedure presented in this chapter is based almost entirely on the principle of virtual work and the previously discussed kinematic influence coefficient relationships and will be addressed in two main sections. The first deals with the first- and second-order kinematics and dynamics and, the second deals, briefly, with the third-order kinematics and the linearized state space model. Also, this chapter deals solely with the basic analytical development (all the components of which have already been discussed) and leaves the more interesting possible applications of this technology for the next chapter.

While the treatment presented here is far more complete in its development and application than any other investigation, and is a generalized (and logical) extension of the one-degree of freedom work of Benedict and Tesar (1971) and the multi-degree of freedom sequential transfer of Freeman (1980), the basic philosophy involved is

not totally unique to the author and his associates. The basic premise is widely accepted and utilized (particularly in the previously described case of the serial manipulator) throughout the community in numerous forms, to various extents, and with differing emphasis. Specific instances of the application of some form of coordinate transfer with regards to control (often misleadingly referred to in the literature as decoupled control) include the resolved motion rate control of Whitney (1969), resolved acceleration control of Luh et al. (1980) and its extension to the linearized equations of motion by Lee and Lee (1984), the active force control scheme of Hewit and Burdess (1981), the hybrid control approach of Raibert and Craig (1981) and, most notably, the operational space work of Khatib (1983) which most closely parallels the work presented here in that it deals with the non-linear second-order geometric transfer question. Other applications dealing more directly with system design than with control include the actuator load, motion investigations of Thomas and Tesar (1982a) and, the effective inertia ellipsoid considerations of Assada and Youcef-Toumi (1983).

The Dynamic Equations

To begin the development of the transfer of generalized coordinates concept a detailed restatement of the situation is beneficial. The procedure consists of four steps and is

presented in terms of first- and second-order system properties.

First, the kinematic influence coefficients (of the dependent system parameters ($\underline{u}$)), and hence, the dynamic model (i.e., the describing equations (S_q)) are not directly obtainable with respect to the generalized coordinates ($\underline{q}$). In other words, one desires the first- and second-order coefficients

$$
\begin{aligned}
[G^u_q] &\equiv \frac{\partial \underline{u}}{\partial \underline{q}} \\
[H^u_{qq}] &\equiv \frac{\partial^2 \underline{u}}{\partial \underline{q} \partial \underline{q}}
\end{aligned}
\tag{3-1}
$$

and the ensuing kinematics (recall equations (2-7) and (2-27))

$$
\begin{aligned}
\underline{u} &= [G^u_q]\dot{\underline{q}} \\
\underline{u} &= [G^u_q]\ddot{\underline{q}} + \dot{\underline{q}}^T[H^u_{qq}]\dot{\underline{q}}
\end{aligned}
\tag{3-2}
$$

and dynamics (recall equation (2-145))

$$
\underline{T}_q = [I^*_{qq}]\ddot{\underline{q}} + \dot{\underline{q}}^T[P^*_{qqq}]\dot{\underline{q}} - \underline{T}^L_q \tag{3-3}
$$

but cannot obtain them directly (i.e., they are unknown).

Second, the describing equations (S_φ) are directly obtainable with respect to some other, initial, set of coordinates ($\underline{\varphi}$). This means that the kinematics

$$
\dot{\underline{u}} = [G^u_\varphi]\dot{\underline{\varphi}} \tag{3-4}
$$

$$\ddot{\underline{u}} = [G^u_\varphi]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^u_{\varphi\varphi}]\dot{\underline{\varphi}}$$

and dynamics

$$\underline{T}_\varphi = [I^*_{\varphi\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[P^*_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} - \underline{T}^L_\varphi \qquad (3\text{-}5)$$

of the system (related to the set ($\underline{\varphi}$)) are known since, the influence coefficients

$$[G^u_\varphi] \equiv \frac{\partial \underline{u}}{\partial \underline{\varphi}} \qquad (3\text{-}6)$$

$$[H^u_{\varphi\varphi}] \equiv \frac{\partial^2 \underline{u}}{\partial \underline{\varphi} \partial \underline{\varphi}}$$

and, hence, the effective model parameters ($[I^*_{\varphi\varphi}]$) and ($[P^*_{\varphi\varphi\varphi}]$) are immediately accessible.

Third, since the desired generalized coordinates ($\underline{q}$) are also known, directly obtainable functions of the initial coordinates ($\underline{\varphi}$), the kinematic influence coefficients

$$[G^q_\varphi] \equiv \frac{\partial \underline{q}}{\partial \underline{\varphi}}$$

$$[H^q_{\varphi\varphi}] \equiv \frac{\partial^2 \underline{q}}{\partial \underline{\varphi} \partial \underline{\varphi}} \qquad (3\text{-}7)$$

relating the desired coordinates ($\underline{q}$) to the initial coordinates ($\underline{\varphi}$) are available. This knowledge allows one to express the desired coordinate velocities ($\dot{\underline{q}}$) and accelerations ($\ddot{\underline{q}}$) as

$$\dot{\underline{q}} = [G^q_\varphi]\dot{\underline{\varphi}} \qquad (3\text{-}8)$$

and

$$\underline{\ddot{q}} = [G_{\varphi}^{q}]\underline{\ddot{\varphi}} + \underline{\dot{\varphi}}^{T}[H_{\varphi\varphi}^{q}]\underline{\dot{\varphi}} \tag{3-9}$$

and, applying the principle of virtual work (recall equation (2-130)), the (effective) load relationship as

$$\underline{T}_{\varphi} = [G_{\varphi}^{q}]^{T}\underline{T}_{q} \tag{3-10}$$

Finally, with the relationships given by equations (3-4), (3-5), (3-8), (3-9) and (3-10), one has all the information necessary to determine the desired unknown parameters in equations (3-2) and (3-3) in terms of the initial, easily obtained, describing equations (S_{φ}). The direct kinematic transfer between the initial ($\underline{\varphi}$) and desired ($\underline{q}$) coordinates (i.e., first- and second-order reverse kinematics solution) is also available.

Kinematic Influence Coefficients

Here, the kinematic relationships of equations (3-2), (3-4), (3-8) and (3-9) are used to obtain the desired influence coefficients. The direct transfer between generalized coordinates is addressed first, and then, the general dependent parameter situation is considered.

Direct kinematic transfer

Consider the case where one wishes to determine the kinematics of the initial coordinates ($\underline{\varphi}(t)$) in terms of the desired coordinate kinematics ($\underline{q}(t)$). This is, in fact, the situation in the resolved rate and acceleration control

schemes of Whitney (1969) and Luh et al. (1980), respectively, where the desired coordinates ($\underline{q}$) are the end-effector motion parameters, and the initial coordinates ($\underline{\varphi}$) are the relative joint angles of a serial manipulator. Recalling the general form of equations (3-2), one wants to determine the coefficients

$$[G_q^{\varphi}] \equiv \frac{\partial \underline{\varphi}}{\partial \underline{q}} \qquad (3\text{-}11)$$

$$[H_{qq}^{\varphi}] \equiv \frac{\partial^2 \underline{\varphi}}{\partial \underline{q} \partial \underline{q}}$$

required (for the generalized parametric form) to evaluate the initial coordinate velocities

$$\underline{\dot{\varphi}} = [G_q^{\varphi}]\underline{\dot{q}} \qquad (3\text{-}12)$$

and accelerations

$$\underline{\ddot{\varphi}} = [G_q^{\varphi}]\underline{\ddot{q}} + \underline{\dot{q}}^T[H_{qq}^{\varphi}]\underline{\dot{q}} \qquad (3\text{-}13)$$

This information (equation (3-11)) is not directly available; however, the reverse relationships of equation (3-7) are. Therefore, from equation (3-8), one has

$$\underline{\dot{\varphi}} = [G_{\varphi}^{q}]^{-1}\underline{\dot{q}} \qquad (3\text{-}14)$$

or, comparing with equation (3-12), the desired first-order kinematic influence coefficient is found to be, simply

$$[G_q^{\varphi}] = [G_{\varphi}^{q}]^{-1} \qquad (3\text{-}15)$$

Now, recalling equation (3-9), the initial coordinate set acceleration vector ($\ddot{\underline{\varphi}}$) can be written as

$$\ddot{\underline{\varphi}} = [G^q_\varphi]^{-1}\ddot{\underline{q}} - [G^q_\varphi]^{-1}\dot{\underline{\varphi}}^T[H^q_{\varphi\varphi}]\dot{\underline{\varphi}} \tag{3-16}$$

Applying the generalized scalar product ($\bullet$) (developed in Appendix A), equation (3-16) becomes

$$\ddot{\underline{\varphi}} = [G^q_\varphi]^{-1}\ddot{\underline{q}} - \dot{\underline{\varphi}}^T([G^q_\varphi]^{-1} \bullet [H^q_{\varphi\varphi}])\dot{\underline{\varphi}} \tag{3-17}$$

or, replacing ($\dot{\underline{\varphi}}$) by equation (3-14),

$$\ddot{\underline{\varphi}} = [G^q_\varphi]^{-1}\ddot{\underline{q}} - \dot{\underline{q}}^T[G^q_\varphi]^{-T}([G^q_\varphi]^{-1} \bullet [H^q_{\varphi\varphi}])[G^q_\varphi]^{-1}\dot{\underline{q}} \tag{3-18}$$

which, when compared with equation (3-13), yields

$$[H^\varphi_{qq}] = -[G^q_\varphi]^{-T}([G^q_\varphi]^{-1} \bullet [H^q_{\varphi\varphi}])[G^q_\varphi]^{-1} \tag{3-19}$$

as the desired second-order influence coefficient (analytically verified, in Appendix C, by comparison with directly obtainable results for a simple manipulator), where the superscript (-T) implies the transpose of the matrix inverse. At this point one should recognize that equations (3-15) and (3-19) allow one to obtain the previously intractable desired influence coefficients in terms of simple directly addressable coefficients related to the initial coordinates ($\underline{\varphi}$) provided the first-order (g,G) function matrix (i.e., Jacobian) of equations (3-7) is square and non-singular. The dependence on well-conditioned Jacobians is an inherent property of coordinate

transformation and will not be dealt with to any significant extent in this work. The use of pseudo-inverses will, however, be discussed with regards to redundant systems in Chapter IV.

Kinemaic transfer of dependent parameters

Suppose the transfer of the dependence of a general dependent system parameter set ($\underline{u}$) from the initial ($\underline{\varphi}$) to the desired ($\underline{q}$) generalized coordinates is required. Here the parametric coefficients of equations (3-1) are desired, allowing the dependent parameter kinematics to be expressed in the form of equations (3-2). Recalling the first of equations (3-4), one has the dependent parameter velocity ($\dot{\underline{u}}$) as

$$\dot{\underline{u}} = [G^u_\varphi]\dot{\underline{\varphi}} \tag{3-20}$$

Now, replacing ($\dot{\underline{\varphi}}$) by equation (3-14) gives

$$\dot{\underline{u}} = [G^u_\varphi][G^q_\varphi]^{-1}\dot{\underline{q}} \tag{3-21}$$

Comparing the result of equation (3-21) with the general form of equation (3-2) yields

$$[G^u_q] = [G^u_\varphi][G^q_\varphi]^{-1} \tag{3-22}$$

as the required first-order coefficient. To investigate the second-order kinematics, recall the second of equation (3-4) for the dependent parameter acceleration

$$\ddot{\underline{u}} = [G^u_\varphi]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^u_{\varphi\varphi}]\dot{\underline{\varphi}} \tag{3-23}$$

Substituting equation (3-16) for the initial coordinate acceleration ($\ddot{\underline{\varphi}}$) into equation (3-23) gives

$$\ddot{\underline{u}} = [G^u_\varphi][G^q_\varphi]^{-1}\ddot{\underline{q}} - [G^u_\varphi][G^q_\varphi]^{-1}\dot{\underline{\varphi}}^T[H^q_{\varphi\varphi}]\dot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[H^u_{\varphi\varphi}]\dot{\underline{\varphi}} \tag{3-24}$$

Now, applying the generalized dot ($\bullet$) operation and combining the quadratic terms yields

$$\ddot{\underline{u}} = [G^u_\varphi][G^q_\varphi]^{-1}\ddot{\underline{q}} + \dot{\underline{\varphi}}^T\{[H^u_{\varphi\varphi}] - ([G^u_\varphi][G^q_\varphi]^{-1} \bullet [H^q_{\varphi\varphi}])\}\dot{\underline{\varphi}} \tag{3-25}$$

Where, finally, replacing ($\dot{\underline{\varphi}}$) with equation (3-14) and comparing the result with the second of equations (3-2), the desired second-order kinematic influence coefficient is

$$[H^u_{qq}] = [G^q_\varphi]^{-T}\{[H^u_{\varphi\varphi}] - ([G^u_\varphi][G^q_\varphi]^{-1} \bullet [H^q_{\varphi\varphi}])\}[G^q_\varphi]^{-1} \tag{3-26}$$

Equations (3-22) and (3-26) again give expressions for the desired coefficients in terms of the easily obtained initial system parameters, and reduce to the direct transfer coefficient equations (3-15) and (3-19) when the dependent parameter ($\underline{u}$) considered is the initial coordinate set ($\underline{\varphi}$) since

$$[G^u_\varphi] = [G^\varphi_\varphi] = [I]_{MxM} \tag{3-27}$$

and

$$[H^u_{\varphi\varphi}] = [H^\varphi_{\varphi\varphi}] = [0] = MxMxM \tag{3-28}$$

Dynamic Model Parameters

One approach (suggested) to the transfer of the dynamic model is to employ the effective load relationship of equation (3-10) along with the previously discussed kinematic transfer results. From equation (3-10), provided the Jacobian is non-singular, the desired generalized loads ($\underline{T}_q$) can be expressed as

$$\underline{T}_q = [G^q_\varphi]^{-T}\underline{T}_\varphi \tag{3-29}$$

or, recalling equation (3-5),

$$\underline{T}_q = [G^q_\varphi]^{-T}([I^*_{\varphi\varphi}]\ddot{\underline{\varphi}} + \dot{\underline{\varphi}}^T[P^*_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} - \underline{T}^L_\varphi) \tag{3-30}$$

Replacing ($\ddot{\underline{\varphi}}$) in equation (3-30) by the result of equation (3-16) gives

$$\begin{aligned}\underline{T}_q = {} & [G^q_\varphi]^{-T}[I^*_{\varphi\varphi}]([G^q_\varphi]^{-1}\ddot{\underline{q}} - [G^q_\varphi]^{-1}\dot{\underline{\varphi}}^T[H^q_{\varphi\varphi}]\dot{\underline{\varphi}}) \\ & + [G^q_\varphi]^{-T}\dot{\underline{\varphi}}^T[P^*_{\varphi\varphi\varphi}]\dot{\underline{\varphi}} - [G^q_\varphi]^{-T}\underline{T}^L_\varphi\end{aligned} \tag{3-31}$$

which, applying the generalized inner product ($\bullet$), combining the quadratic terms and substituting the result of equation (3-14) for ($\dot{\underline{\varphi}}$), yields

$$\begin{aligned}\underline{T}_q = {} & [G^q_\varphi]^{-T}[I^*_{\varphi\varphi}][G^q_\varphi]^{-1}\ddot{q} \\ & + \dot{q}^T[G^q_\varphi]^{-T}\{([G^q_\varphi]^{-T} \bullet [P^*_{\varphi\varphi\varphi}]) - ([G^q_\varphi]^{-T}[I^*_{\varphi\varphi}][G^q_\varphi]^{-1} \\ & \bullet [H^q_{\varphi\varphi}])\}[G^q_\varphi]^{-1}\dot{q} - \sum_{i=1}^{N}([{}^iG^u_\varphi][G^q_\varphi]^{-1})^T\, {}^i\underline{T}^u\end{aligned} \tag{3-32}$$

Finally, comparing this result with the general form of equation (3-3), one has the effective generalized loads ($\underline{T}_q$)

seen by the desired generalized coordinates ($\underline{q}$) in terms of parameters directly obtainable from the initial coordinates ($\underline{\varphi}$) as

$$\underline{T}^q = [I^*_{qq}]\ddot{\underline{q}} + \dot{\underline{q}}^T[P^*_{qqq}]\dot{\underline{q}} - \underline{T}^L_q \tag{3-33}$$

where

$$[I^*_{qq}] = [G^q_\varphi]^{-T}[I^*_{\varphi\varphi}][G^q_\varphi]^{-1} \tag{3-34}$$

is the desired generalized inertia matrix,

$$\begin{aligned}[P^*_{qqq}] = [G^q_\varphi]^{-T}\{([G^q_\varphi]^{-T} \bullet [P^*_{\varphi\varphi\varphi}]) \\ - ([I^*_{qq}] \bullet [H^q_{\varphi\varphi}])\}[G^q_\varphi]^{-1}\end{aligned} \tag{3-35}$$

is the inertia power array yielding the velocity related effective loads, and the desired effective load due to all applied loads (${}^i\underline{T}^u$, i = 1,2,...,n) is

$$\underline{T}^L_q = [G^q_\varphi]^{-T}\ \underline{T}^L_\varphi \tag{3-36}$$

Alternately, the dynamic model can be transfered by direct use of the generic (for constant mass systems) equations (2-139), (2-141) and (2-145) once the desired (transferred) kinematic influence coefficients have been obtained. For example, consider the determination of the desired effective load ($\underline{T}^L_q$) due to applied system loads. Recalling equation (2-125), one has

$$\underline{T}^L_q = \sum_{i=1}^{N} [{}^iG^u_q]^T\ {}^i\underline{T}^u \tag{3-37}$$

or, recalling equation (3-22) for the kinematic transfer

$$\underline{T}^{L}_{q} = \sum_{i=1}^{N} ([{}^{i}G^{u}_{\varphi}][G^{q}_{\varphi}]^{-1})^{T} \; {}^{i}\underline{T}^{u} \qquad (3\text{-}38)$$

Now, to demonstrate the equality of the results of equations (3-36) and (3-38), rewrite equation (3-38) as

$$\underline{T}^{L}_{q} = \sum_{i=1}^{N} [G^{q}_{\varphi}]^{-T}[{}^{i}G^{u}_{\varphi}]^{T} \; {}^{i}\underline{T}^{u} \qquad (3\text{-}39)$$

which becomes

$$\underline{T}^{L}_{q} = [G^{q}_{\varphi}]^{-T} \sum_{i=1}^{N} [{}^{i}G^{u}_{\varphi}]^{T} \; {}^{i}\underline{T}^{u} \qquad (3\text{-}40)$$

Finally, recalling from the general form of equation (2-125) that

$$\underline{T}^{L}_{\varphi} = \sum_{i=1}^{N} [{}^{i}G^{u}_{\varphi}]^{T} \; {}^{i}\underline{T}^{u} \qquad (3\text{-}41)$$

equation (3-40) can be written as

$$\underline{T}^{L}_{q} = [G^{q}_{\varphi}]^{-T} \; \underline{T}^{L}_{\varphi} \qquad (3\text{-}42)$$

illustrating the equality of the results of the two model transfer techniques.

As previously mentioned, the full power and freedom that this transfer of generalized coordinates affords one when investigating the design, analysis, and control of complicated mechanical systems will become apparent through the study of the possible applications presented in Chapter IV. Here it will only be pointed out that, with the

particular resultant form of the kinematic and dynamic models presented in this work, the generalized scalar product (•) is the key realization and operation.

The Linearized Equations

The consideration of the transfer of the linearized dynamic equations and the resulting state space model will be addressed in two parts. The first deals simply with the direct transfer of the third-order kinematics (allowing one to determine, for example, the voltage input required to obtain a specified nominal hand trajectory), while the second discusses a possible state model resulting from a partial transfer of coordinates.

Kinematics

Recalling the general form of equation (2-28) and the second set of equations (2-44), the third-order time state ($\dddot{\underline{q}}$) of the desired coordinates ($\underline{q}$) can be expressed, in terms of the initial coordinates ($\underline{\varphi}$) (e.g., the relative joint angles of a serial manipulator), as

$$\dddot{\underline{q}} = \frac{\partial \ddot{\underline{q}}}{\partial \ddot{\underline{\varphi}}}\dddot{\underline{\varphi}} + \frac{\partial \ddot{\underline{q}}}{\partial \dot{\underline{\varphi}}}\ddot{\underline{\varphi}} + \frac{\partial \ddot{\underline{q}}}{\partial \underline{\varphi}}\dot{\underline{\varphi}} \qquad (3\text{-}43)$$

or

$$\dddot{\underline{q}} = [G^{q}_{\varphi}]\dddot{\underline{\varphi}} + \dot{\underline{\varphi}}^{T}([H^{q}_{\varphi\varphi}] + 2[H^{q}_{\varphi\varphi}]^{T})\ddot{\underline{\varphi}} + (\dot{\underline{\varphi}}^{T}[D^{q}_{\varphi\varphi\varphi}]\dot{\underline{\varphi}})\dot{\underline{\varphi}} \qquad (3\text{-}44)$$

where the influence coefficients are considered known up to the third order. Now, solving equation (3-44) for the initial coordinate's jerk (i.e., $\dddot{\underline{\varphi}}$), one has

$$\dddot{\underline{\varphi}} = [G^q_\varphi]^{-1}(\dddot{\underline{q}} - \frac{\partial \ddot{\underline{q}}}{\partial \dot{\underline{\varphi}}}\ddot{\underline{\varphi}} - \frac{\partial \ddot{\underline{q}}}{\partial \underline{\varphi}}\dot{\underline{\varphi}}) \qquad (3\text{-}45)$$

or

$$\dddot{\underline{\varphi}} = [G^q_\varphi]^{-1}(\dddot{\underline{q}} - \ddot{\underline{\varphi}}^T([H^q_{\varphi\varphi}] + 2[H^q_{\varphi\varphi}]^T)\dot{\underline{\varphi}} - (\dot{\underline{\varphi}}^T[D^q_{\varphi\varphi\varphi}]\dot{\underline{\varphi}})\dot{\underline{\varphi}}) \qquad (3\text{-}46)$$

At this point the previously derived expressions for ($\underline{\varphi}$) and ($\underline{\varphi}$) (i.e., the results of equations (3-14) and (3-18)) could be substituted into equation (3-46), the result could then be manipulated and compared with the general form of equation (2-44), ultimately yielding

$$[D^\varphi_{qqq}] = f([G^q_\varphi],[H^q_{\varphi\varphi}],[D^q_{\varphi\varphi\varphi}]) \qquad (3\text{-}47)$$

However, the manipulation required to obtain a nice result (such as that afforded by the inner product for the second-order coefficient) for equation (3-46) is excessive and not required for the task at hand. That task, again, is the evaluation of equation (2-240) relating the required actuator voltages to the nominal joint motions ($\underline{\varphi}(t)$) resulting from the desired end-effector trajectory ($\underline{q}(t)$). Therefore, to obtain the required voltages ($\underline{v}$), all one needs to do is solve equations (3-14), (3-16) or (3-18), and (3-45) for ($\dot{\underline{\varphi}}$), ($\ddot{\underline{\varphi}}$) and ($\dddot{\underline{\varphi}}$) in terms of the specified hand motion ($\underline{q}(t)$) and then substitute the results into equation (2-240). While the computational effort involved here (in

what might be referred to as resolved-jerk control) is considerable, the procedure is straight-forward with the required voltages determined off-line and only their numerical values stored for real-time recall.

State Space Equations

The presentation here (as with the state space treatment of Chapter II), while conceptually generic, deals specifically with the case of the d-c motor actuated serial manipulator. In this case, since the actuators are generally associated with the joint motions and cannot be practically located at the end-effector, only a partial transfer of states (i.e., generalized coordinate perturbation) is considered. Here, the dependence on the linkage-related states (i.e., the joint coordinate perturbations $(\delta\underline{\varphi})$, $(\delta\dot{\underline{\varphi}})$, $(\delta\ddot{\underline{\varphi}})$) is transfered to the end-effector motion perturbations (i.e., $(\delta\underline{q})$, $(\delta\dot{\underline{q}})$ and $(\delta\ddot{\underline{q}})$) while the actuator locations remain unaltered.

Recalling equation (2-241), one has

$$\begin{pmatrix} \delta\dot{\underline{\varphi}} \\ \hline \delta\ddot{\underline{\varphi}} \\ \hline \delta\dot{\underline{i}} \end{pmatrix} = \left[\begin{array}{c|c|c} F_{11} & F_{12} & F_{13} \\ \hline F_{21} & F_{22} & F_{23} \\ \hline F_{31} & F_{32} & F_{33} \end{array}\right] \begin{pmatrix} \delta\underline{\varphi} \\ \hline \delta\dot{\underline{\varphi}} \\ \hline \delta\underline{i} \end{pmatrix} + \left[\begin{array}{c} G_{11} \\ \hline G_{21} \\ \hline G_{31} \end{array}\right] \delta\underline{V} \qquad (3\text{-}48)$$

where the partition matrices (F_{ij} and G_{ij}) are as defined by equation (2-242) through (2-252). Now referring to the form of the kinematic equation (2-5), (2-11) and (2-28) and applying the kinematic transfer concept, one can express the

joint related perturbations in terms of the hand perturbations as

$$\delta\underline{\varphi} = [G^q_\varphi]^{-1}\delta\underline{q} \tag{3-49}$$

$$\delta\underline{\dot{\varphi}} = [G^q_\varphi]^{-1}\delta\underline{\dot{q}} - [G^q_\varphi]^{-1}\left(\frac{\partial\,\underline{\dot{q}}}{\partial\,\underline{\varphi}}\right)[G^q_\varphi]^{-1}\delta\underline{q} \tag{3-50}$$

and

$$\delta\underline{\ddot{\varphi}} = [G^q_\varphi]^{-1}\delta\underline{\ddot{q}} - [G^q_\varphi]^{-1}\left(\frac{\partial\,\underline{\ddot{q}}}{\partial\,\underline{\dot{\varphi}}}\right)[G^q_\varphi]^{-1}\delta\underline{\dot{q}}$$
$$- [G^q_\varphi]^{-1}\left\{\left(\frac{\partial\,\underline{\ddot{q}}}{\partial\,\underline{\varphi}}\right) - \left(\frac{\partial\,\underline{\ddot{q}}}{\partial\,\underline{\dot{\varphi}}}\right)[G^q_\varphi]^{-1}\left(\frac{\partial\,\underline{\dot{q}}}{\partial\,\underline{\varphi}}\right)\right\}[G^q_\varphi]^{-1}\delta\underline{q} \tag{3-51}$$

Finally, substituting equations (3-49), (3-50) and (3-51) into the initial state space representation of equation (3-48) gives

$$\begin{pmatrix} \delta\underline{\dot{q}} \\ \hline \delta\underline{\ddot{q}} \\ \hline \delta\underline{\dot{i}} \end{pmatrix} = \left[\begin{array}{c|c|c} \hat{F}_{11} & \hat{F}_{12} & \hat{F}_{13} \\ \hline \hat{F}_{21} & \hat{F}_{22} & \hat{F}_{23} \\ \hline \hat{F}_{31} & \hat{F}_{32} & \hat{F}_{33} \end{array}\right] \begin{pmatrix} \delta\underline{q} \\ \hline \delta\underline{\dot{q}} \\ \hline \delta\underline{i} \end{pmatrix} + \left[\begin{array}{c} \hat{G}_{11} \\ \hline \hat{G}_{21} \\ \hline \hat{G}_{31} \end{array}\right] \delta\underline{V} \tag{3-52}$$

as the desired state model, where

$$\hat{G}_{i1} = F_{i1} \quad , \ i=1,2,3$$

$$\hat{F}_{ij} = F_{ij} \quad , \ j=1,2,3$$

$$\hat{F}_{21} = \left\{\left(\frac{\partial\,\underline{\ddot{q}}}{\partial\,\underline{\varphi}}\right) - \left(\frac{\partial\,\underline{\ddot{q}}}{\partial\,\underline{\dot{\varphi}}}\right)[G^q_\varphi]^{-1}\left(\frac{\partial\,\underline{\dot{q}}}{\partial\,\underline{\varphi}}\right)\right.$$
$$\left. + [G^q_\varphi](F_{21} - F_{22}[G^q_\varphi]^{-1}\left(\frac{\partial\,\underline{\dot{q}}}{\partial\,\underline{\varphi}}\right))\right\}[G^q_\varphi]^{-1}$$

$$\hat{F}_{22} = \left\{\left(\frac{\partial\,\underline{\ddot{q}}}{\partial\,\underline{\dot{\varphi}}}\right) + [G^q_\varphi]F_{22}\right\}[G^q_\varphi]^{-1} \tag{3-53}$$

$$\hat{F}_{23} = [G^q_\varphi]F_{23}$$

$$\hat{F}_{31} = -F_{32}[G_{\varphi}^{q}]^{-1}\left(\frac{\partial \dot{\underline{q}}}{\partial \underline{\varphi}}\right)[G_{\varphi}^{q}]^{-1}$$
$$\hat{F}_{32} = F_{32}[G_{\varphi}^{q}]^{-1}$$
$$\hat{F}_{33} = F_{33}$$

with the partials relating the desired ($\underline{q}$) and initial ($\underline{\varphi}$) coordinates given by the general form of equations (2-14), (2-35) and (2-43).

Equivalently, one could first derive expressions for the effective joint loads ($\underline{T}_{\varphi}$) in terms of the desired coordinate ($\underline{q}$) kinematics (Thomas and Tesar, 1982b), and then follow the linearization procedure of Chapter II to obtain the state model of equation (3-52). This method simply reverses the order in which the transfer of coordinates takes place and does not affect the results.

Alternately, one could work with the original (i.e., initial) state model of equation (3-48) and simply use equations (3-49) and (3-50) to obtain the states (i.e., $\delta\underline{\varphi}$ and $\delta\dot{\underline{\varphi}}$) in terms of measured end-effector perturbations (i.e., $\delta\underline{q}$ and $\delta\dot{\underline{q}}$). This is the more common approach (generally taken by the investigators cited at the beginning of this chapter) but does not yield much additional insight with regards to system control. Here, the state model (on which the controller design is based) is still referenced to the joint coordinates ($\underline{\varphi}$), whereas the transfer procedure actually affords one a new state model (directly referenced to the desired end-effector motion parameters ($\underline{q}$)) on which to base the controller design. While the transferred model

approach may (or may not) yield a better controller design (since, after all, as the joint perturbations go to zero so to do those at the end-effector (for a relatively-rigid-link device)) it at least allows one to view the system from a different point of view.

CHAPTER IV

APPLICATION OF GENERALIZED COORDINATE TRANSFORMATION TO DYNAMIC MODELING AND CONTROL--A UNIFIED APPROACH

The transfer of generalized coordinate technique, in conjunction with the kinematic influence coefficient based model formulation for which it was developed, gives one the ability to consider an incredible variety of linkages (and their interactions with one another) with a single unified approach. The approach involves the relatively simple modeling of serial kinematic chains, the application (single or multiple) of the transformation equations, and if necessary (e.g., when dealing with redundancies) the inclusion of classical optimization techniques, such as the method of Lagrange multipliers. This chapter, then, deals with the application of this unified approach to the analysis of a representative selection of single and multiple linkage systems illustrating its full power and scope. In addition, the utility of (and reason for) the general descriptive notation presented in the Introduction is more fully demonstrated as a consequence of the applications.

Single Closed-loop Mechanisms

Consider an unconstrained general M-degree of freedom serial manipulator (i.e., open kinematic chain) operating relative to an N-dimensional space, with ($M \geq N$). Further,consider that the free end ($\underline{e}$) of this chain is connected to ground by a constraint of degree R (relative to the same N-space). The resulting closed-loop mechanism is then seen (refer to Fig. 4-1) to be of mobility (M-R). Now, the task at hand is to analyze this mechanism (for a given configuration) from any desired, (M-R)-dimensional, set of generalized coordinates.

The first step, as is always the case, is to model the system as if it were an unconstrained open kinematic chain. Therefore, using the modeling procedure developed in Chapter II for the serial manipulator one immediately (directly) obtains

$$[G^e_\varphi] = N \times M \tag{4-1}$$

$$[H^e_{\varphi\varphi}] = N \times M \times M \tag{4-2}$$

$$[I^*_{\varphi\varphi}] = M \times M \tag{4-3}$$

$$[P^*_{\varphi\varphi\varphi}] = M \times M \times M \tag{4-4}$$

and

$$\underline{T}^L_\varphi = M \times 1 \tag{4-5}$$

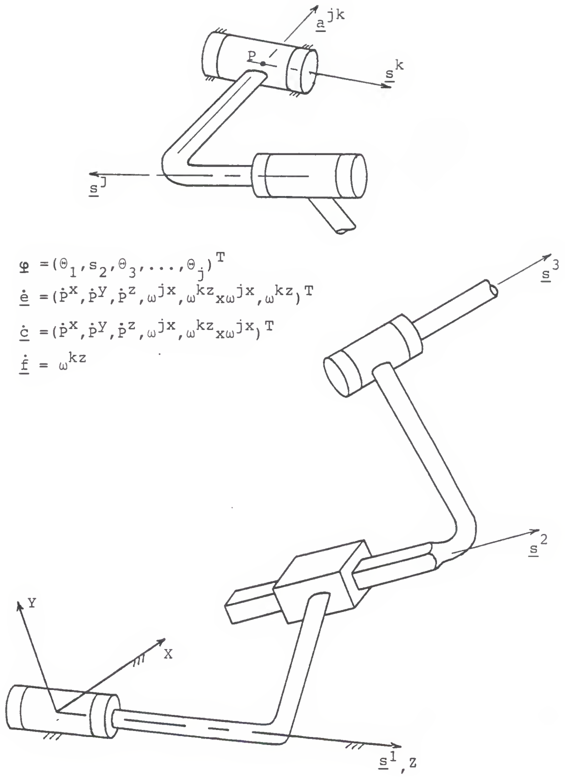


Figure 4-1. Single closed-loop mechanism

where

$$\underline{e} = (e_1, e_2, \ldots, e_N)^T \tag{4-6}$$

describes the motion parameters associated with the free end, and the initial generalized coordinates

$$\underline{\varphi} = (\varphi_1, \varphi_2, \ldots \varphi_M)^T \tag{4-7}$$

are associated with the joint freedoms of the open chain.

Now, with M≥N, (M-N) of the initial coordinates ($\underline{\varphi}$) must be involved in any set of generalized coordinates since one can constrain (or control) at most N of the free end freedoms (i.e., R≤N). As a consequence of this redundant-like situation (i.e., M≥N) the dependent parameter set ($\underline{e}$) must be augmented by (M-N) of the initial coordinates. Here, for convenience, take the first (M-N) components of ($\underline{\varphi}$), defined as ($\underline{\alpha}$), to be the parameters chosen to augment ($\underline{e}$), yielding

$$\underline{w} = \begin{pmatrix} \underline{\alpha} \\ -- \\ \underline{e} \end{pmatrix} = \begin{pmatrix} (M-N) \times 1 \\ ----- \\ N \times 1 \end{pmatrix} = M \times 1 \tag{4-8}$$

as an intermediate set ($\underline{w}$) of desired coordinates.

For future reference, the remaining N components of the initial set ($\underline{\varphi}$) are defined by the vector ($\underline{\beta}$), giving

$$\underline{\varphi} = \begin{pmatrix} \underline{\alpha} \\ -- \\ \underline{\beta} \end{pmatrix} = \begin{pmatrix} (M-N) \times 1 \\ ----- \\ N \times 1 \end{pmatrix} = M \times 1 \tag{4-9}$$

and recalling that one will eventually constrain R of the parameters ($\underline{e}$), define

$$\underline{e} = \begin{pmatrix} \underline{f} \\ -- \\ \underline{c} \end{pmatrix} = \begin{pmatrix} (N-R) \times 1 \\ ----- \\ R \times 1 \end{pmatrix} = N \times 1 \tag{4-10}$$

where the set ($\underline{f}$) describes the remaining freedoms of ($\underline{e}$) after the R constraints, which allow no motion of the parameters ($\underline{c}$)(i.e., $\dot{\underline{c}}=\ddot{\underline{c}}=0$), are applied. Again, as with the partitioning of ($\underline{\varphi}$), the sequential ordering in the set ($\underline{e}$) of the components making up ($\underline{f}$) and ($\underline{c}$) is done merely for convenience in discussing the general case (i.e., it is not necessary).

Returning to the augmented set ($\underline{w}$), one is now able to express the first-order kinematic influence coefficients relating the intermediate coordinates ($\underline{w}$) to the initial coordinates ($\underline{\varphi}$) as

$$[G^{w}_{\varphi}] = \left[\begin{array}{c|c} I & O \\ \hline G^{e}_{\alpha} & G^{e}_{\beta} \end{array}\right] = \left[\begin{array}{c|c} (M-N) \times (M-N) & (M-N) \times N \\ \hline N \times (M-N) & N \times N \end{array}\right] = M \times M \tag{4-11}$$

with (from equation (4-1))

$$[G^{e}_{\alpha} \mid G^{e}_{\beta}] = [G^{e}_{\varphi}] \tag{4-12}$$

Recalling that

$$\dot{\underline{w}} = [G^{w}_{\varphi}]\dot{\underline{\varphi}} \tag{4-13}$$

one sees that the first (M-N) rows of equation (4-11) merely state that

$$\dot{\underline{\alpha}} = \dot{\underline{\alpha}} \tag{4-14}$$

Correspondingly, the second-order influence coefficient matrix becomes (recalling how the dimensions of the super- and subscripts give the dimension of the result)

$$[H^{W}_{\varphi\varphi}] = \left[\begin{array}{c|c} H^{W}_{\alpha\alpha} & H^{W}_{\alpha\beta} \\ \hline H^{W}_{\beta\alpha} & H^{W}_{\beta\beta} \end{array}\right] = \left[\begin{array}{c|c} Mx(M-N)x(M-N) & Mx(M-N)x(N) \\ \hline Mx(N)x(M-N) & Mx(N)x(N) \end{array}\right] = MxMxM \tag{4-15}$$

where the first (M-N) planes are given by

$$[H^{\alpha}_{\varphi\varphi}] = [0] = (M-N) \times M \times M \tag{4-16}$$

since

$$\underline{\ddot{\alpha}} = \underline{\ddot{\alpha}} \tag{4-17}$$

and the last N planes ($[H_{\varphi\varphi}]$) are given by equation (4-2).

Next, recalling the direct transfer equation (3-15), and noting the block partitioned form of equation (4-11), one has

$$[G^{\varphi}_{w}] = [G^{w}_{\varphi}]^{-1} = \left[\begin{array}{c|c} [I] & [0] \\ \hline -[G^{e}_{\beta}]^{-1}[G^{e}_{\alpha}] & [G^{e}_{\beta}]^{-1} \end{array}\right] = \left[\begin{array}{c|c} (M-N)x(M-N) & (M-N)xN \\ \hline Nx(M-N) & NxN \end{array}\right] \tag{4-18}$$

for the intermediate first-order coefficients, which recalling equations (4-8), (4-9) and (4-10), yields

$$[G^{\varphi}_{w}] = \left[\begin{array}{c|c} I & O \\ \hline G^{\beta}_{\alpha} & G^{\beta}_{e} \end{array}\right] = \left[\begin{array}{c|c|c} I & \multicolumn{2}{c}{O} \\ \hline G^{\beta}_{\alpha} & G^{\beta}_{f} & G^{\beta}_{c} \end{array}\right] = MxM \tag{4-19}$$

Notice that the parameters ($\underline{\alpha}$) are still independent but the parameters ($\underline{\beta}$) are now dependent on the full intermediate set ($\underline{w}$). Now, recalling equation (3-19), the intermediate second-order coefficient is

$$[H^{\varphi}_{ww}] = -[G^{w}_{\varphi}]^{-T}([G^{w}_{\varphi}]^{-1} \bullet [H^{w}_{\varphi\varphi}])[G^{w}_{\varphi}]^{-1} = MxMxM \tag{4-20}$$

where the first (M-N) planes are, again, MxM null matrices (since ($\underline{\alpha}$) is independent) and the last N planes give

$$[H^{\beta}_{ww}] = \left[\begin{array}{c|c} H^{\beta}_{\alpha\alpha} & H^{\beta}_{\alpha e} \\ \hline H^{\beta}_{e\alpha} & H^{\beta}_{ee} \end{array}\right] = \left[\begin{array}{c|c} Nx(M-N)x(M-N) & Nx(M-N)xN \\ \hline NxNx(M-N) & NxNxN \end{array}\right] \tag{4-21}$$

or, recalling equation (4-10),

$$[H^{\beta}_{ww}] = \left[\begin{array}{c|cc} H^{\beta}_{\alpha\alpha} & H^{\beta}_{\alpha f} & H^{\beta}_{\alpha c} \\ \hline H^{\beta}_{f\alpha} & H^{\beta}_{ff} & H^{\beta}_{fc} \\ H^{\beta}_{c\alpha} & H^{\beta}_{cf} & H^{\beta}_{cc} \end{array}\right] = NxMxM \tag{4-22}$$

Note that, due to the form of ($[G_{\varphi}]^{-1}$) and ($[H_{\varphi\varphi}]$), to save unneccessary multiplication by zero, the last N planes of equation (4-20) can be (and in actuality are) obtained from

$$[H^{\beta}_{ww}] = -[G^{w}_{\varphi}]^{-T}([G^{e}_{\beta}]^{-1} \bullet [H^{e}_{\varphi\varphi}])[G^{w}_{\varphi}]^{-1} \tag{4-23}$$

Now, refering to equations (3-33) through (3-36), one can obtain the intermediate dynamic equations as

$$\underline{T}_{w} = [I^{*}_{ww}]\underline{\ddot{w}} + \underline{\dot{w}}^{T}[P^{*}_{www}]\underline{\dot{w}} - \underline{T}^{L}_{w} \tag{4-24}$$

or

$$\begin{pmatrix} {}_w\underline{T}_\alpha \\ -- \\ {}_w\underline{T}_f \\ -- \\ {}_w\underline{T}_c \end{pmatrix} = [I^*_{ww}] \begin{pmatrix} \ddot{\underline{\alpha}} \\ -- \\ \ddot{\underline{f}} \\ -- \\ \ddot{\underline{c}} \end{pmatrix} + (\dot{\underline{\alpha}}^T, \dot{\underline{f}}^T, \dot{\underline{c}}^T)[P^*_{www}] \begin{pmatrix} \dot{\underline{\alpha}} \\ -- \\ \dot{\underline{f}} \\ -- \\ \dot{\underline{c}} \end{pmatrix} - \underline{T}^L_w \qquad (4\text{-}25)$$

where

$$[I^*_{ww}] = [G^w_\varphi]^{-T}[I^*_{\varphi\varphi}][G^w_\varphi]^{-1} \qquad (4\text{-}26)$$

$$[P^*_{www}] = [G^w_\varphi]^{-T}\{([G^w_\varphi]^{-T} \bullet [P^*_{\varphi\varphi\varphi}]) - ([I^*_{ww}] \bullet [H^w_{\varphi\varphi}])\}[G^w_\varphi]^{-1} \qquad (4\text{-}27)$$

and

$$\underline{T}^L_w = [G^w_\varphi]^{-T}\, \underline{T}^L_\varphi \qquad (4\text{-}28)$$

Again, there are computational simplifications that can be made in equations (4-26), (4-27) and (4-28) due, not only to the special forms of the component matrices in the equations, but also to the fact that ($\dot{\underline{c}}=\ddot{\underline{c}}=\underline{0}$) here. These simplifications are left to the reader to investigate; but it needs to be stressed that while the kinematics of the set ($\underline{\alpha}$) remain independent the dynamic effects of the system on ($\underline{\alpha}$) have been altered by the active transfer of the N parameters ($\underline{\beta}$) to the N parameters ($\underline{e}$). In other words, given the same kinematic state (defined by say, $\underline{e}(t)$) and applied loading condition, the intermediate effective load (${}_w\underline{T}_\alpha$) on the generalized coordinates ($\underline{\alpha}$) is not the same as the initial effective load (${}_\varphi\underline{T}_\alpha$). Perhaps the easiest way to demonstrate this is to note the difference in the effect of the applied loads. Recalling equations (4-18) and (4-19), equation (4-28) gives

$${}_w\underline{T}^L_\alpha = {}_\varphi\underline{T}^L_\alpha + [{}_wG^T_\alpha]^T\, {}_\varphi\underline{T}^L_\beta \qquad (4\text{-}29)$$

where the preceding subscripts indicate which generalized coordinate set the parameter is related to (i.e., the preceding subscript gives additional information concerning the independent parameters).

Now, returning to equations (4-25) through (4-28), in light of the R constraints ($\underline{c}$) it is convenient to define a new intermediate parameters set ($\underline{y}$) given by the remaining (M-R) free coordinates as

$$\underline{y} = \begin{pmatrix} \underline{\alpha} \\ - - \\ \underline{f} \end{pmatrix} = \begin{pmatrix} (M-N)x1 \\ - - - - \\ (N-R)x1 \end{pmatrix} = (M-R)x1 \qquad (4-30)$$

With this notation one can rewrite equation (4-25) as

$$\begin{pmatrix} \underline{T}_y \\ - - \\ \underline{T}_c \end{pmatrix} = \lfloor I^*_{wy} \mid I^*_{wc} \rfloor \begin{pmatrix} \ddot{\underline{y}} \\ - - \\ \ddot{\underline{c}} \end{pmatrix} + (\dot{\underline{y}}^T, \dot{\underline{c}}^T) \left[\begin{array}{c|c} P^*_{wyy} & P^*_{wyc} \\ \hline P^*_{wcy} & P^*_{wcy} \end{array} \right] \begin{pmatrix} \dot{\underline{y}} \\ - - \\ \dot{\underline{c}} \end{pmatrix} - \begin{pmatrix} \underline{T}^L_y \\ - - \\ \underline{T}^L_c \end{pmatrix} \qquad (4-31)$$

since the parameters ($\underline{c}$) are now fixed (i.e., $\underline{c}=\underline{c}=\underline{0}$). Note that equation (4-31) not only gives the effective loads ($\underline{T}_y$) required to drive the (M-R)-degree of freedom mechanism from the coordinate set ($\underline{y}$) (for some specified kinematic state $\underline{y}$, $\dot{\underline{y}}$, $\ddot{\underline{y}}$), but also gives the resulting constraint forces ($\underline{T}_c$) (if desired). Additionally, the first of equations (4-31) can be used to address the effects of (and on) non-rigid bearings. The similarity between the closure approach taken here and the 7-R mechanism approach to the reverse position solution of the general 6-R manipulator of Duffy and Crane (1980) and the Uicker et al.(1964) iterative

analysis method should also be noted. Also, the results of equation (4-31) are verified (in Appendix D) (for the simple case of the planar 5-Bar) by comparison with the results of a direct dyad-based analysis.

Now, if the second intermediate coordinate set ($\underline{y}$) is the desired set of (M-R) generalized coordinates from which one wishes to actuate the mechanism, one is done. If, however, the desired set of coordinates ($\underline{q}$) differs from ($\underline{y}$), one must (in part) first obtain the intermediate dynamics. From equation (4-31), one has

$$\underline{T}_y = [I^*_{yy}]\ddot{\underline{y}} + \dot{\underline{y}}^T[P^*_{yyy}]\dot{\underline{y}} - \underline{T}^L_y \tag{4-32}$$

where the first (M-R) rows of ($[I^*_{wy}]$) gives ($[I^*_{yy}]$) and the first (M-R) planes of ($[P^*_{wyy}]$) gives ($[P^*_{yyy}]$). Next, one needs to determine the influence coefficients ($[G^q_y]$) and ($[H^q_{yy}]$) relating the desired set ($\underline{q}$) to ($\underline{y}$).

Assuming, for now, that ($\underline{q}$) is a subset of the parameter sets ($\underline{\alpha}$), ($\underline{\beta}$) and ($\underline{f}$) one merely needs to recognize that

$$[G^\beta_y] = [G^\beta_\alpha \,|\, G^\beta_f] = N \times (M\text{-}R) \tag{4-33}$$

from the previous results of equation (4-19), and from equation (4-22) that

$$[H^\beta_{yy}] = \left[\begin{array}{c|c} H^\beta_{\alpha\alpha} & H^\beta_{\alpha f} \\ \hline H^\beta_{f\alpha} & H^\beta_{ff} \end{array}\right] = N x (M\text{-}R) x (M\text{-}R) \tag{4-34}$$

and since the coordinates ($\underline{y}$) are now considered to be the generalized coordinates (i.e., independent) that

$$[G^y_y] = [I] = (M-R) \times (M-R) \tag{4-35}$$

and

$$[H^y_{yy}] = [0] = (M-R) \times (M-R) \times (M-R) \tag{4-36}$$

Now, following the technique (which will be more rigorously illustrated in the next application) implied by the construction of the influence coefficient matrices ($[G^w_\varphi]$) and ($[H^w_{\varphi\varphi}]$) relating the augmented set ($\underline{w}$) to the initial set ($\underline{\varphi}$), one can simply piece together the required influence coefficients

$$[G^q_y] = (M-R) \times (M-R) \tag{4-37}$$

and

$$[H^q_{yy}] = (M-R) \times (M-R) \times (M-R) \tag{4-38}$$

from the known relationships of equations (4-33) through (4-36) for any (M-R) desired components ($\underline{q}$) of the set ($\underline{\alpha}$, $\underline{\beta}$, $\underline{f}$). Note that judicious selection of ($\underline{\alpha}$) will simplify this task.

Finally, the desired describing equations are obtained from a second application of the generalized transfer equations as

$$\begin{aligned} [G^y_q] &= [G^q_y]^{-1} \\ [H^y_{qq}] &= -[G^q_y]^{-T}([G^q_y]^{-1} \bullet [H^q_{yy}])[G^q_y]^{-1} \end{aligned} \tag{4-39}$$

and

$$\underline{T}_q = [I^*_{qq}]\underline{\ddot{q}} + \underline{\dot{q}}^T[P^*_{qqq}]\underline{\dot{q}} - \underline{T}^L_q \tag{4-40}$$

where

$$[I^*_{qq}] = [G^q_y]^{-T}[I^*_{yy}][G^q_y]^{-1}$$

$$[P^*_{qqq}] = [G^q_y]^{-T}\{([G^q_y]^{-T} \bullet [P^*_{yyy}]) - ([I^*_{qq}] \bullet [H^q_{yy}])\}[G^q_y]^{-1}$$

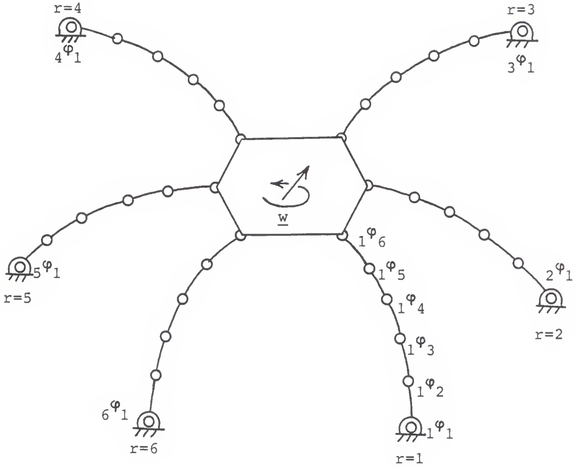
$$\underline{T}^L_q = [G^q_y]^{-T}\ \underline{T}^L_y \tag{4-41}$$

Thus, (by the double application of the transfer technique) one is able to model any single-loop mechanism (with the actuators located at any set ($\underline{q}$) of the linkage joints) by simply determining the joint-referenced model of an open kinematic chain. Now, if the actuators are associated with coordinates other than the joints ($\underline{\varphi}$, $\underline{f}$) themselves, to obtain the required coefficients ($[G^q_y]$) and ($[H^q_{yy}]$) one must take care to use the general dependent parameter kinematic transfer equations to carry the desired parameter along throughout the complete transfer process. This, then actually allows one to obtain the model for (any) generalized coordinate set. In fact, this procedure avails itself to the analysis of the two main types of special-case single-loop linkages. Those being; mechanisms containing unspecified independent coordinates, and mechanisms having one or more redundant constraints. These two cases are discussed briefly in Appendix D in terms of the spatial slider-crank and Bricard mechanisms, respectively.

Multi-loop Parallel Mechanisms

The procedure for the modeling of multi-loop parallel mechanisms is not as one-directional as that for the single-loop case, and therefore, is not addressed in all-inclusive general terms. This is not to say that a much more general class of parallelism cannot be addressed by the same procedure, but simply that there is no single general outline for the infinite variety of conceivable parallel devices. Here, the primary concern will be for linkages of the class illustrated by the generalized Stewart platform represented in Fig. 4-2, where the figure is intended to be viewed, for example, as six general six-degree of freedom manipulators all grasping the same object (or having a common end-effector. Also, by class what is meant is that each branch (or leg) has the same number of joints as the mechanism has freedoms (i.e., in the single-loop case $M = N$), and that there are no coupling links between legs (which would require non-general techniques to obtain the related influence coefficients).

Now, addressing the six-degree of freedom mechanism of Fig. 4-2 (i.e., the generalized Stewart platform), suppose that one wishes to model (or control) this device from any collection of six of the thirty-six of joint freedoms (i.e., generalized coordinates where, as in the single-loop



Desired Generalized Coordinates

$$\underline{q} = ({}_1\varphi_1, {}_2\varphi_1, {}_3\varphi_1, {}_4\varphi_1, {}_5\varphi_1, {}_6\varphi_1)^T$$

Initial Generalized Coordinates

$${}_r\underline{\varphi} = ({}_r\varphi_1, {}_r\varphi_2, {}_r\varphi_3, {}_r\varphi_4, {}_r\varphi_5, {}_r\varphi_6)^T \; ; \; r = 1,2,\ldots,6$$

Figure 4-2. Generalized Stewart platform

situation, one notes that the joint motion parameters are not the only possible generalized coordinates). Again, the first step is to directly model the system with respect to some initial set of coordinates. Here the approach is to model each leg (r=1,2,...,6) with respect to its own joint coordinate set (${}_r\underline{\varphi}$) as if the other legs did not exist. This yields, using the serial manipulator procedure of Chapter II, six sets of describing equations, with the r^{th} one given by

$$({}_r S_\varphi) = [{}_r G^w_\varphi], [{}_r H^w_{\varphi\varphi}], [{}_r I^*_{\varphi\varphi}], [{}_r P^*_{\varphi\varphi\varphi}], {}_r \underline{T}^L_\varphi \tag{4-42}$$

where ($\underline{w}$) is an intermediate coordinate set associated with the six parameters describing the motion of the platform, and where the mass of the platform, along with any load applied directly to the platform, is neglected (although it could be included in the model of one of the legs).

Next, apply the transfer equations to each leg yielding

$$[{}^r G^\varphi_w] = [{}_r G^w_\varphi]^{-1} \tag{4-43}$$

$$[{}^r H^\varphi_{ww}] = -[{}_r G^w_\varphi]^{-T}([{}_r G^w_\varphi]^{-1} \bullet [{}_r H^w_{\varphi\varphi}])[{}_r G^w_\varphi]^{-1} \tag{4-44}$$

$$[{}^r I^*_{ww}] = [{}_r G^w_\varphi]^{-T}[{}_r I^*_{\varphi\varphi}][{}_r G^w_\varphi]^{-1} \tag{4-45}$$

$$\begin{aligned}[{}^r P^*_{www}] = [{}_r G^w_\varphi]^{-T}\{&([{}_r G^w_\varphi]^{-T} \bullet [{}_r P^*_{\varphi\varphi\varphi}]) \\ &-([{}^r I^*_{ww}] \bullet [{}_r H^w_{\varphi\varphi}])\}[{}_r G^w_\varphi]^{-1}\end{aligned} \tag{4-46}$$

and

$${}^r \underline{T}^L_w = [{}_r G^w_\varphi]^{-T} \; {}_r \underline{T}^L_\varphi \tag{4-47}$$

for the model of each leg referenced to the intermediate coordinates ($\underline{w}$). Now, combining the effects of each leg and including the inertial effects of the platform and the resolved vector of loads applied directly to the platform, one has

$$[I^*_{ww}] = [I_{ww}] + \sum_{r=1}^{6} [{}^rI^*_{ww}]$$

$$[P^*_{www}] = [P_{www}] + \sum_{r=1}^{6} [{}^rP^*_{www}] \tag{4-48}$$

and

$$\underline{T}^L_w = \begin{pmatrix} {}^6\underline{f}^c \\ - - - \\ \underline{m}^{67} \end{pmatrix} + \sum_{r=1}^{6} {}^r\underline{T}^L \tag{4-49}$$

where

$$[I_{ww}] = \left[\begin{array}{ccc|c} M^{67} & & 0 & \\ & M^{67} & & \\ 0 & & M^{67} & 0 \\ \hline & 0 & & [II^{67}] \end{array}\right] \tag{4-50}$$

and where the first three planes of ($[P_{www}]$) are 6x6 null matrices, and

$$[P_{www}]_{l;;} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & [P_{lww}] \end{array}\right] \quad , \; l = 4,5,6 \tag{4-51}$$

where

$$[P_{4ww}] = [\underline{0} \mid \underline{I}^z \mid -\underline{I}^y] = 3x3$$

$$[P_{5ww}] = [-\underline{I}^z \mid \underline{0} \mid \underline{I}^x] = 3x3 \tag{4-52}$$

$$[P_{6ww}] = [\underline{I}^y \mid -\underline{I}^x \mid \underline{0}] = 3x3$$

with

$$[II^{67}] = [\underline{I}^x \mid \underline{I}^y \mid \underline{I}^z] \tag{4-53}$$

since (by choice)

$$\underline{w} \equiv \begin{pmatrix} {}^6\underline{v}^c \\ \hline \underline{\omega}^{67} \end{pmatrix} \tag{4-54}$$

At this stage, having obtained the description of the total mechanism referenced to the intermediate parameters ($\underline{w}$), one may proceed to transfer the model to any desired set ($\underline{q}$) of six of the thirty-six joint freedoms. Here, for illustration, assume that the desired coordinates are the base joints of each leg, or

$$\underline{q} = ({}_1\varphi_1,\ {}_2\varphi_1,\ \ldots,\ {}_6\varphi_1)^T \tag{4-55}$$

yielding, for the first- and second-order influence coefficients relating the desired coordinates ($\underline{q}$) to the intermediate coordinates ($\underline{w}$) (from equations (4-43) and (4-44))

$$[G_w^q] = \begin{bmatrix} [{}^1G_w^{\varphi}]_{1;} \\ \hline [{}^2G_w^{\varphi}]_{1;} \\ \hline \vdots \\ \hline [{}^6G_w^{\varphi}]_{1;} \end{bmatrix} \tag{4-56}$$

and

$$[H_{ww}^q] = \begin{bmatrix} [{}^1H_{ww}^{\varphi}]_{1;;} \\ \hline [{}^2H_{ww}^{\varphi}]_{1;;} \\ \hline \vdots \\ \hline [{}^6H_{ww}^{\varphi}]_{1;;} \end{bmatrix} \tag{4-57}$$

Before continuing, note that (in general) if

$$q_n = r\varphi_m \tag{4-58}$$

then

$$[G^q_w]_{n;} = [{}^rG^\varphi_w]_{m;} \tag{4-59}$$

and

$$[H^q_{ww}]_{n;;} = [{}^rH^\varphi_{ww}]_{m;;} \tag{4-60}$$

Finally, applying the transfer equations, one obtains for the desired describing equations

$$\underline{T}_q = [I^*_{qq}]\ddot{\underline{q}} + \dot{\underline{q}}^T[P^*_{qqq}]\dot{\underline{q}} - \underline{T}^L_q \tag{4-61}$$

where

$$[I^*_{qq}] = [G^q_w]^{-T}[I^*_{ww}][G^q_w]^{-1}$$

$$[P^*_{qqq}] = [G^q_w]^{-T}\{([G^q_w]^{-T} \bullet [P^*_{www}]) - ([I^*_{qq}] \bullet [H^q_{ww}])\}[G^q_w]^{-1} \tag{4-62}$$

$$\underline{T}^L_q = [G^q_w]^{-T}\ \underline{T}^L_w$$

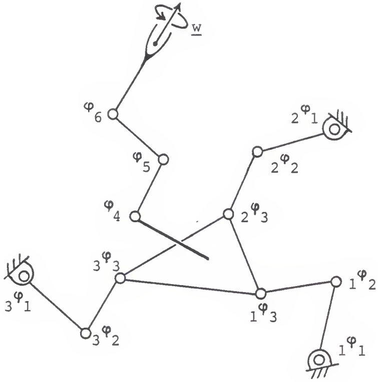
with

$$[G^w_q] = [G^q_w]^{-1} \tag{4-63}$$

and

$$[H^w_{qq}] = -[G^q_w]^{-T}([G^q_w]^{-1} \bullet [H^q_{ww}])[G^q_w]^{-1} \tag{4-64}$$

Here, the dynamic model of an extremely complicated multi-input spatial mechanism has been derived in terms of relatively simple serial manipulator relationships, again,



Desired Generalized Coordinates

$$\underline{q} = ({}_1\varphi_1, {}_2\varphi_1, {}_3\varphi_1, \varphi_4, \varphi_5, \varphi_6)^T$$

Initial Generalized Coordinates

$$_r\underline{\varphi} = ({}_r\varphi_1, {}_r\varphi_2, {}_r\varphi_3, \varphi_4, \varphi_5, \varphi_6)^T \; ; \; r = 1,2,3$$

Figure 4-3. Parallel-serial mechanism

via a double application of the generalized coordinate transfer equations. In fact, the hybrid parallel-serial mechanism (Fig. 4-3) treated by Sklar (1984) can be modeled using the same procedure. To see this, simply include (φ_4), (φ_5) and (φ_6) in each of the three initial sets (${}_r\underline{\varphi}$, r=1,2,3). Include the inertial effects of links (3), (4), (5) and (6) in the direct (initial) modeling of one (and only one) branch, and note that (φ_4), (φ_5) and (φ_6) must (also) be included in the desired coordinate set ($\underline{q}$). Also note, the reverse position problem (i.e., specify ($\underline{w}$) and determine ($\underline{q}$)) for both devices can be addressed using the analysis techniques of Duffy (1980) and Duffy and Crane (1980). The results of this parallel-mechanism analysis are verified (in Appendix E) by simulation of a simple three-degree of freedom planar device.

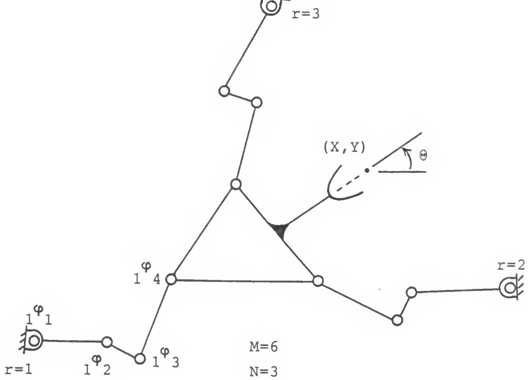
Redundant Systems

Consider a general M-degree of freedom system ($\underline{q} \varepsilon$ R^M) operating relative to an N-dimensional space ($\mathbb{R}^N$). Further, consider that there exists a superabundance of kinematically independent inputs ($\underline{q}$) with which to obtain the most general motion possible ($\underline{u} \varepsilon$ $\mathbb{R}^N$) for a free unconstrained body ($\underline{u}$) residing in that space (i.e., M>N). This superabundance of kinematically independent inputs (refered to here and in the general literature as redundancy) is perhaps most easily illustrated by the

general serial manipulator with more than six inputs (i.e., M>N=6). The situation is of course exhibited by a much broader class of systems. Take for example, the planar, multi-loop, parallel six-degree of freedom mechanism shown in Fig. 4-4. Here, there are six independent inputs (all of which must be controlled) available to drive the system according to the most general motion specification possible (i.e., $x=x(t)$, $y=y(t)$ and $\Theta=\Theta(t)$). Therefore, there are three redundant inputs (three more than required) in this particular robotic device. Alternately, this type of redundancy exists when only a partial specification (e.g., translation but not orientation) of the end-effector trajectory of a general six-degree of freedom serial manipulator is made, or when the trajectory is obtainable with a subset of the available inputs. Now, without considering the specifics of the particular system in question, or the justification of the optimization criteria (i.e., cost function) employed, the application of the transfer of generalized coordinates to the modeling of redundant mechanisms will be discussed.

Here, it is assumed that the position of the system is known and that the dynamic model of the system with respect to the desired coordinates ($\underline{q}$) has been established, yielding

$$\dot{\underline{u}} = [G^u_q]\dot{\underline{q}} \tag{4-65}$$



$$\underline{q} = ({}_1\varphi_1, {}_2\varphi_1, {}_3\varphi_1, {}_1\varphi_2, {}_2\varphi_2, {}_3\varphi_2)^T \in \mathbb{R}^M$$

$$\underline{u} = (X, Y, \Theta)^T \in \mathbb{R}^N$$

Figure 4-4. Six-degree of freedom planar device

$$\ddot{\underline{u}} = [G^u_q]\ddot{\underline{q}} + \dot{\underline{q}}[H^u_{qq}]\dot{\underline{q}} \tag{4-66}$$

and

$$\underline{T}_q = [I^*_{qq}]\ddot{\underline{q}} + \dot{\underline{q}}^T[P^*_{qqq}]\dot{\underline{q}} - \underline{T}^L_q \tag{4-67}$$

In viewing equations (4-65), (4-66) and (4-67), one sees that the relationships are unique for specified values of the desired coordinates ($\underline{q}$) velocities ($\dot{\underline{q}}$) and accelerations ($\ddot{\underline{q}}$). Unfortunately (from the modeling and control point of view) however, the desired coordinate kinematics (i.e., $\dot{\underline{q}}$ and $\ddot{\underline{q}}$) are not uniquely determined by specification of the desired kinematics (i.e., $\dot{\underline{u}}$ and $\ddot{\underline{u}}$) of the controlled variable ($\underline{u}$).

Therefore, the first step is to determine (M-N) additional constraining relationships between the coordinates ($\underline{q}$) and ($\underline{u}$). Here, one typically decides to optimize (instead of arbitrarily specifying (M-N) kinematic relationships among the ($\underline{q}$)) some performance criteria involving the superabundant coordinates ($\underline{q}$) subject to the required kinematic constraints (e.g., equation (4-65)) relating the controlled parameter ($\underline{u}$) to ($\underline{q}$). Once the performance criteria is established one can use any of a number of optimization techniques (e.g., the classical Lagrange-multiplier approach) to determine the additional constraints between ($\underline{q}$) and ($\underline{u}$). The end result of this being the reverse of equation (4-65), or

$$\dot{\underline{q}} = [G^q_u]\dot{\underline{u}} \tag{4-68}$$

where

$$[G^q_u] = [G^u_q]^{-R} = MxN \qquad (4\text{-}69)$$

is the pseudo (right) inverse of the unique (NxM) jacobian of equation (4-65). For example, if one wishes to minimize the kinetic energy of the system (i.e., equation (2-140)), subject to the constraint of equation (4-65), one obtains (Whitney, 1969)

$$[G^u_q]^{-R} = [I^*_{qq}]^{-1}[G^u_q]^T([G^u_q][I^*_{qq}]^{-1}[G^u_q]^T)^{-1} \qquad (4\text{-}70)$$

for the desired pseudo inverse.

Having established the pseudo inverse given by equation (4-69), one can (though it is not necessary) now apply the coordinate transfer technique to obtain the reverse of equation (4-66) as

$$\underline{\ddot{q}} = [G^q_u]\underline{\ddot{u}} + \underline{\dot{u}}^T[H^q_{uu}]\underline{\dot{u}} \qquad (4\text{-}71)$$

where

$$[H^q_{uu}] = -[G^u_q]^{-RT}([G^u_q]^{-R} \bullet [H^u_{qq}])[G^u_q]^{-R} \qquad (4\text{-}72)$$

Now, finally, substituting the results of equations (4-68) and (4-70) into equation (4-67) yields

$$\underline{T}_q = [I^*_{qu}]\underline{\ddot{u}} + \underline{\dot{u}}^T[P^*_{quu}]\underline{\dot{u}} - [G^u_q]^{-RT}\underline{T}_u \qquad (4\text{-}73)$$

as the effective generalized loads at the desired

coordinates required to obtain the specified controlled variable kinematics in accordance with the given performance criteria, where

$$[I^*_{qu}] = [I^*_{qq}][G^u_q]^{-R} = [G^u_q]^T([G^u_q][I^*_{qq}]^{-1}[G^u_q]^T)^{-1} = MxN \tag{4-74}$$

and

$$[P^*_{quu}] = [G^u_q]^{-RT}\{[P^*_{qqq}] - ([I^*_{qu}] \bullet [H^u_{qq}])\}[G^u_q]^{-R} = MxNxN \tag{4-75}$$

Here (see also Thomas and Tesar, 1982a, equation (3.35) for a non-redundant manipulator), it should be noted that the effective loads at ($\underline{q}$) are expressed directly in terms of the controlled variable kinematics resulting in what could be refered to as a partial transfer of coordinates.

Overconstrained Systems

The devices considered in this section can also be thought of as redundant in that the redundancy comes from a superabundance of kinematically (dependent) inputs, as illustrated by the three-degree of freedom, four (and six)-input systems of Fig. 4-5. This situation is dramatically different from the previous case where the superabundance was of kinematically (independent) inputs. In that case, the only kinematic invariant is the controlled variable itself, and the joint loads are uniquely determined after the remaining system kinematics are selected (via optimization). Here, once the kinematics of the controlled

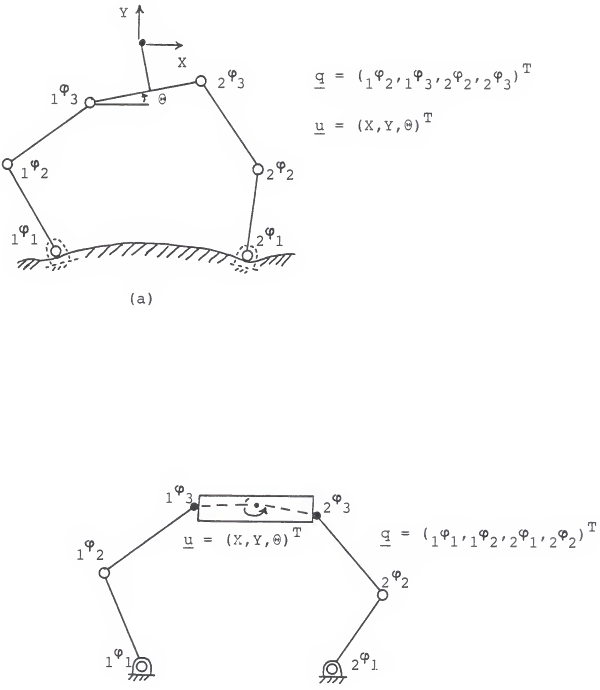


Figure 4-5. Overconstrained systems. a) Walking Machine ; b) Multi-fingered hand/Cooperating Manipulators

variable are specified, the kinematic state of the entire system is uniquely determined (i.e., all system parameters are kinematically invariant with respect to the controlled variable). The result of this invariance is an antagonistic (rather than cooperative) situation, where the load generated by one input is geometrically (rather than through, dynamic, inertial effects) transferred to the others. As illustrated by the planar examples of Fig. 4-5, this is exactly the situation encountered when dealing with the coordinated action of multiple open-chain systems, such as walking machines, multi-fingered hands and cooperating manipulators.

Here, without loss of generality, the specific case of the two-fingered hand (Fig. 4-5b) is addressed. In addition to the analytic generality afforded by this example, it also allows the reader to immediately verify (by simple experiment) the absolute necessity of the direct actuation of the four parameters

$$\underline{q} = ({}_1\varphi_1,\ {}_1\varphi_2,\ {}_2\varphi_1,\ {}_2\varphi_2)^T \tag{4-76}$$

to even hold the object ($\underline{u}$) stationary (much less maintain some kinematic trajectory). Therefore, one must determine a balanced loading among the four actuators while, at the same time maintaining the desired kinematics of the dependent controlled variable ($\underline{u}$).

The first step, once the kinematics of ($\underline{u}$) are specified, is to obtain the model of the system (where the contact points are viewed, instantaneously, as pin joints)

referenced to the intermediate generalized coordinates ($\underline{u}$). To do this one follows exactly the same procedure as represented in the parallel mechanism section (equations (4-42) through (4-60) with $\underline{w}=\underline{u}$), yielding the unique results

$$T_u = [I^*_{uu}]\ddot{\underline{u}} + \dot{\underline{u}}^T[P^*_{uuu}]\dot{\underline{u}} - \underline{T}^L_u = 3\text{x}1 \tag{4-77}$$

$$[G^q_u] = \begin{bmatrix} [{}^1G^\varphi_u]_{;1} \\ \\ [{}^1G^\varphi_u]_{;2} \\ \\ [{}^2G^\varphi_u]_{;1} \\ \\ [{}^2G^\varphi_u]_{;2} \end{bmatrix} = 4\text{x}3 \tag{4-78}$$

and

$$[H^q_{uu}] = \begin{bmatrix} [{}^1H^\varphi_{uu}]_{1;;} \\ - - - - - \\ [{}^1H^\varphi_{uu}]_{2;;} \\ - - - - - \\ [{}^2H^\varphi_{uu}]_{1;;} \\ - - - - - \\ [{}^2H^\varphi_{uu}]_{2;;} \end{bmatrix} = 4\text{x}3\text{x}3 \tag{4-79}$$

Now, to distribute (via the Lagrange-multiplier approach) the required generalized load ($\underline{T}_u$) between the actuators (q) while maintaining the desired kinematics one may use any of a number of objective functions (i.e., optimizing criteria), but must use the constraint relationship

$$\underline{c} = [G^q_u]^T\underline{T}_q - \underline{T}_u = \underline{0} \tag{4-80}$$

since, because the kinematics are unique, ($\underline{T}_u$) is unique. If, for example, one decides to minimize the sum of the squares of the desired torques (i.e., the square of the p=2 norm), then the objective function is given by

$$f = \underline{T}_q^T \underline{T}_q \tag{4-81}$$

yielding, with the constraint ($\underline{c}$), the Lagrangian

$$\bar{f} = \underline{T}_q^T \underline{T}_q + \underline{\lambda}^T([G_u^q]^T \underline{T}_q - \underline{T}_u) \tag{4-82}$$

Setting the partials with respect to the independent variables to zero gives

$$\left(\frac{\partial \bar{f}}{\partial \underline{T}_q}\right)^T = 2\underline{T}_q + [G_u^q]\underline{\lambda} = \underline{0} = 4x1 \tag{4-83}$$

and

$$\left(\frac{\partial \bar{f}}{\partial \underline{\lambda}}\right) = [G_u^q]^T \underline{T}_q - \underline{T}_u = \underline{0} = 3x1 \tag{4-84}$$

resulting in

$$\underline{T}_q = [G_u^q]([G_u^q]^T[G_u^q])^{-1}\underline{T}_u \tag{4-85}$$

which, when compared with equation (4-80), gives the pseudo (left) inverse transpose

$$[G_u^q]^{-LT} = [G_q^u]^T = [G_u^q]([G_u^q]^T[G_u^q])^{-1} = 4x3 \tag{4-86}$$

Finally, applying the transfer equations once more (i.e., substituting equation (4-77) into equation (4-85)), one obtains the desired distributed load ($\underline{T}_q$) as

$$\underline{T}_q = [I^*_{qu}]\underline{u} + \underline{u}^T[P^*_{quu}]\underline{u} - \underline{T}^L_q \qquad (4\text{-}87)$$

with

$$[I^*_{qu}] = [G^q_u]^{-LT}[I^*_{uu}] = 4x3 \qquad (4\text{-}88)$$

$$[P^*_{quu}] = ([G^q_u]^{-LT} \bullet [P^*_{uuu}]) = 4x3x3 \qquad (4\text{-}89)$$

and

$$\underline{T}^L_q = [G^q_u]^{-LT}\ \underline{T}^L_u \qquad (4\text{-}90)$$

where, again, only a partial transformation of coordinates is employed. As with the solution for the redundant case (equation (4-73)), the partial transfer is employed because the resulting equations are in the desired form; that is, the (desired) required generalized loads are expressed directly in terms of the controlled variable kinematics.

CHAPTER V

CONCLUSIONS

The major thrust of this work has been the development of a single general method which allows one to investigate the dynamics of (almost) any rigid-link mechanism (or system of mechanisms) from any desired set of generalized coordinates. The method requires the determination of an initial dynamic model(s) in terms of the simplest linkage possible, the open-loop kinematic chain. Once this elementary model(s) is obtained the principle of virtual work and the kinematic constraints (i.e., kinematic influence coefficients) relating the initial generalized coordinates to any other set of generalized coordinates allow the determination of the model(s) referenced to the other (desired) set of coordinates. Thus, one is able to determine the dynamic model of extremely complicated linkage systems (or systems referenced to an obscure set of generalized coordinates) in terms of easily obtained elementary open-chain model parameters. The approach was shown to enable the dynamic analysis of

1. The general serial manipulator in terms of the end-effector coordinates (or any other set of generalized coordinates).

2. Single-loop mechanisms referenced to any generalized coordinate set.
3. Multi-loop parallel-input mechanisms.
4. Systems with a superabundance of kinematically independent inputs.
5. Systems with a superabundance of kinematically dependent inputs.

Because of the dependence of the unified approach on the chosen generic model form and the relative joint angle referenced open-loop kinematic chain, an exhaustive treatment of these elementary, yet fundamental, concepts was given. In addition, a new and descriptive notation was developed as an illustrative aid in the analysis of the systems discussed. The graphic separation of dependent and independent system parameters afforded by this notation is extremely beneficial when dealing with the transference of system dependence among various sets of generalized coordinates, and is the primary reason for its introduction.

The main drawback of the overall approach is the need to invert the Jacobian relating the desired generalized coordinates to the initial generalized coordinates. This is not only computationally demanding, but is, of course, also subject to singularities. While the issue of singularities was not addressed in the work, it is noted that a singular Jacobian does not mean that the system cannot, in general, attain the desired motion. What is meant is that a reduction in the allowable motion space has occurred, and if

the desired motion is not in that reduced space, it cannot be obtained. Again, while the issue of singularities is not dealt with in general, the treatment of the Bricard mechanism (Appendix D) addresses a special case of this phenomena.

As comprehensive as this work is, it is merely a foundation on which to base the more complete treatment required for real-time dynamic compensation and control of robotic devices. To achieve the ultimate goal of precision operation under load will require considerably more than the rigid-link model presented here. Of high priority for the immediate future (and addressable by the concepts developed in this work) are the issues of

1. Metrology (Behi, 1985)
   The determination of the system parameters required for the quantitative model (e.g., link dimensions, mass and stiffness parameters, etc.).
2. Local Compensation for In-Plane Deformation (Tesar and Kamen, 1983)
   Both feedforward and feedback compensation for link and actuator deformations occurring in the plane locally addressable by each individual actuator.
3. Global Compensation for General Deformations (Fresonke, 1985)
   Feedforward and feedback compensation for dynamic and load induced end-effector deflection due to general link deformations.

4. Sufficient-Model Determination (Wander, 1985)
   Investigation of the controlling equations to determine the minimum model (as well as the most efficient computational scheme) sufficient for real-time dynamic compensation.
5. Control of Redundant Systems (Tesar and Kamen, 1983)
   Determination of algorithms appropriate for the control of systems with a superabundance of kinematically independent inputs envisioned for the separation of large and small motion priorities, as well as, for enhanced operating volume and dexterity.
6. Control of Overconstrained Systems (Orin and McGhee, 1981)
   In-depth investigation of the antagonistic nature of systems with a superabundance of kinematically dependent inputs (e.g., walking machines, multi-fingered end-effectors, cooperating robots) aimed at the determination of the ideal set of generalized coordinates for control.
7. Hybrid Control (Raibert and Craig, 1981)
   Investigation of system control in terms of a set of generalized coordinates consisting of a mix of kinematic and dynamic parameters.

In conclusion, while there is much to be done, it is suggested that this work removes (to a large extent) one of the most difficult tasks facing the design (or control)

engineer. That is, the determination of the basic mathematical model representing the system dynamics. It is hoped that the removal of this burden (and the additional insight obtained by viewing the system from different sets of generalized coordinates) will encourage one to be more creative and investigate the possibility of systems that might otherwise be thought unattainable.

APPENDIX A

DEVELOPMENT AND DEFINITION OF GENERALIZED SCALAR (•) PRODUCT OPERATOR FUNDAMENTAL TO DYNAMIC MODELING AND TRANSFER OF COORDINATES

1. Quadratic Operation of a Matrix on a 3-dimension Array

Given

$$[D] = M \times N \quad \text{Matrix}$$

$$[C] = N \times M \times M \quad \text{Array}$$

Define

$$[D]^T[C][D] \equiv \begin{bmatrix} [D]^T\ [C]_{1;;}[D] \\ \hline [D^T]\ [C]_{2;;}[D] \\ \hline \vdots \\ \hline [D]^T\ [C]_{N;;}[D] \end{bmatrix}$$

$$= N \times M \times M \quad \text{Array}$$

Where

$$[C]_{n;;} = n^{th} \text{ plane of } [C]$$

$$= M \times M \quad \text{Matrix}$$

2. Quadratic Operation of a Vector on a 3-dimension Array

Given

$$\underline{b} = M \text{ component column vector}$$

Define

$$\underline{b}^T[C]\underline{b} \equiv \begin{pmatrix} \underline{b}^T[C]_{1;;}\ \underline{b} \\ \underline{b}^T[C]_{2;;}\ \underline{b} \\ \vdots \\ \underline{b}^T[C]_{n;;}\ \underline{b} \end{pmatrix}$$

$$= M \times 1 \quad \text{Vector}$$

3. Vector Multiplication of Quadratic Result

Given

$$[A] = P \times N \text{ matrix}$$

$$[A]_{p;} = p^{th} \text{ row of } [A] = ([A]_{p;1}, [A]_{p;2}, \cdots, [A]_{p;N})$$

Then

$$[A]_{p;}(\underline{b}^T[C]\underline{b}) = [A]_{p;1}\underline{b}^T[C]_{1;;}\underline{b} + [A]_{p;2}\underline{b}^T[C]_{2;;}\underline{b} + \cdots + [A]_{p;N}\underline{b}^T[C]_{N;;}\underline{b}$$

$$= \underline{b}^T[A]_{p;1}[C]_{1;;}\underline{b} + \underline{b}^T[A]_{p;2}[C]_{2;;}\underline{6} + \cdots + \underline{b}^T[A]_{p;N}[C]_{N;;}\underline{b}$$

$$= \underline{b}^T\left(\sum_{n=1}^{N} [A]_{p;n}[C]_{n;;}\right)\underline{b}$$

$$= \text{scalar}$$

Define Operator (•) "Dot"

$$([A]_{p;} \bullet [C]) \equiv \left(\sum_{n=1}^{N} [A]_{p;n}[C]_{n;;}\right) = N \times N \text{ matrix}$$

= scalar multiplication of planes followed by a summation of the resulting planes

4. Matrix Multiplication of Quadratic Result using (•) operator

$$[A](\underline{b}^T[C]\underline{b}) = \begin{pmatrix} \underline{b}^T([A]_{1;} \bullet [C])\underline{b} \\ \underline{b}^T([A]_{2;} \bullet [C])\underline{b} \\ \\ \underline{b}^T([A]_{p;} \bullet [C])\underline{b} \end{pmatrix}$$

$$= \underline{b}^T([A] \bullet [C])\underline{b}$$

$$= P \times 1 \text{ vector}$$

<u>Where</u>

$$([A] \bullet [C]) = P \times M \times M \text{ array}$$

<u>Therefore</u>

$$([{}^{i}G^{u}_{q}]^{T}[{}^{i}M^{u}])\dot{\underline{q}}^{T}[{}^{i}H^{u}_{qq}]\dot{\underline{q}} = \dot{\underline{q}}^{T}(([{}^{i}G^{u}_{q}]^{T}[{}^{i}M^{u}]) \bullet [{}^{i}H^{u}_{qq}])\dot{\underline{q}}$$

APPENDIX B

SECOND-ORDER TRANSFER FOR A SIMPLE MANIPULATOR

Here the specific case of the wrist-partitioned manipulator shown in Fig. B-1 and treated by Hollerbach and Sahar (1983) is addressed. In this class of manipulator (i.e., wrist-partitionable) the orientation of the end-effector is completely specified by (and completely specifies) the last three joint positions. This allows the inverse kinematics of the first three joints ($\underline{\varphi}$) to be obtained in terms of the cartesian coordinates of a wrist-point ($\underline{P}$), where the wrist-point ($\underline{P}$) is determined directly from the position and orientation of the end-effector. (i.e., independent of the last three joints). As a result of this partitioned solution capability the kinematic influence coefficients relating the joint coordinates ($\underline{\varphi}$) to the wrist coordinates ($\underline{P}$) can be determined directly (and simply), without the need to employ the transfer of generalized coordinate approach. Therefore, the wrist-partitioned manipulator is ideally suited to serve as vehicle to verify (algebraically) the results of the generalized coordinate transformation approach to the inverse kinematics problem (specifically, the transferred (h,H) function result).

Assuming that the kinematics of the wrist-point ($\underline{P}$) are known, one can express the acceleration vector ($\ddot{\underline{\varphi}}$) in the general form of equation (2-27) as

$$\ddot{\underline{\varphi}} = [G^{\varphi}_{p}]\ddot{\underline{P}} + \dot{\underline{P}}^{T}[H^{\varphi}_{pp}]\dot{\underline{P}}$$

Here, for brevity, only the acceleration of the first joint will be considered. From equation (B-1), one has

$$\ddot{\Theta}_1 = [G^{\varphi}_{p}]_{1;}\ddot{\underline{P}} + \dot{\underline{P}}^{T}[H^{\varphi}_{pp}]_{1;;}\dot{\underline{P}}$$

The primary objective is now to show that

$$([H^{\varphi}_{pp}]_{1;;})_{\text{transferred}} = ([H^{\varphi}_{pp}]_{1;;})_{\text{direct}}$$

where, from equation (3-19)

$$([H^{\varphi}_{pp}]_{1;;})_{t} = -[G^{P}_{\varphi}]^{-T}([G^{P}_{\varphi}]^{-1} \bullet [H^{P}_{\varphi\varphi}])_{1;;}[G^{P}_{\varphi}]^{-1}$$

Referring to Table B-1, one has that

$$[G^{P}_{\varphi}] = [\underline{g}^{P}_{1} \mid \underline{g}^{P}_{2} \mid \underline{g}^{P}_{3}]$$

which yields

$$[G^{\varphi}_{P}] = [G^{P}_{\varphi}]^{-1} = \begin{bmatrix} -\dfrac{s\Theta_1}{r} & \dfrac{c\Theta_1}{r} & 0 \\ \dfrac{c\Theta_1\, c\Theta_{2+3}}{a_{23}s\Theta_3} & \dfrac{s\Theta_1\, c\Theta_{2+3}}{a_{23}s\Theta_3} & \dfrac{s\Theta_{2+3}}{a_{23}s\Theta_3} \\ -\dfrac{rc\Theta_1}{a_{23}a_{34}s\Theta_3} & -\dfrac{rs\Theta_1}{a_{23}s_{34}s\Theta_3} & -\dfrac{(1)_{P}z}{a_{23}a_{34}s\Theta_3} \end{bmatrix}$$

where, $s\Theta_{1+2}=\sin(\Theta_1 + \Theta_2)$, etc. Applying the generalized dot operator gives

$$([G^P_\varphi]^{-1} \bullet [H^P_{\varphi\varphi}])_{1;;} = ([G^P_\varphi]^{-1})_{1;} \bullet [H^P_{\varphi\varphi}]$$

$$= (g^1_p)^T \bullet [H^P_{\varphi\varphi}] \qquad \text{(B-7)}$$

$$= -\frac{1}{r}\begin{bmatrix} 0 & {}^{(1)}p^z & a_{34}s\Theta_{2+3} \\ {}^{(1)}p^z & 0 & 0 \\ a_{34}s\Theta_{2+3} & 0 & 0 \end{bmatrix}$$

where $[H_{\varphi\varphi}]$ is given in Table B-1. Finally, performing the remaining multiplications one has that

$$([H^\varphi_{PP}]_{1;;})_t = \frac{1}{r^2}\begin{bmatrix} s(2\Theta_1) & -c(2\Theta_1) & 0 \\ -c(2\Theta_1) & -s(2\Theta_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(B-8)}$$

or, for the acceleration of joint one

$$\ddot{\Theta}_1 = \frac{1}{r}[-s\Theta_1 \;\; c\Theta_1 \;\; 0]\underline{\ddot{P}} + \frac{1}{r^2}\,\underline{\dot{P}}^T\begin{bmatrix} s(2\Theta_1) & -c(2\Theta_1) & 0 \\ -c(2\Theta_1) & s(2\Theta_1) & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{\dot{P}} \qquad \text{(B-9)}$$

where recognizing that

$${}^{(1)}P^z c\Theta_{2+3} - r\, s\Theta_{2+3} = -a_{23}s\Theta_3 \;,\; r = {}^{(1)}P^x \qquad \text{(B-10)}$$

aids considerably in the derivation of equation (B-8). Now, from Fig. B-1, one has the two simple relationships

$$\begin{aligned} rc\Theta_1 &= P^x \\ rs\Theta_1 &= P^y \end{aligned} \qquad \text{(B-11)}$$

Taking the derivative of equation (B-11) with respect to time yields

$$\begin{aligned} -r\dot{\Theta}_1 s\Theta_1 + \dot{r}c\Theta_1 &= \dot{P}^x \\ r\dot{\Theta}_1 c\Theta_1 + \dot{r}s\Theta_1 &= \dot{P}^y \end{aligned} \qquad \text{(B-12)}$$

which can be solved to give

$$\dot{r} = \dot{P}^{x}c\Theta_1 + \dot{P}^{y}s\Theta_1 \qquad \text{(B-13)}$$

and

$$r\Theta_1 = -P^{x}s\Theta_1 + P^{y}c\Theta_1 \qquad \text{(B-14)}$$

Differentiating equation (B-14) with respect to time, and substituting equation (B-13) for ($\dot{r}$) and the solution of equation (B-14) for ($\dot{\Theta}_1$) in the resulting expression gives

$$\ddot{\Theta}_1 = \frac{1}{r}(\ddot{P}^{y}c\Theta_1 - \ddot{P}^{x}s\Theta_1) + \frac{2}{r^2}(\dot{P}^{x}c\Theta_1 + \dot{P}^{y}s\Theta_1)(\dot{P}^{y}c\Theta_1 - \dot{P}^{x}s\Theta_1) \qquad \text{(B-15)}$$

which yields the same result as that obtained using the transfer of coordinate approach (i.e., equation (B-9), thereby verifying the sought after relationship of equation (B-3).

For an interesting example of the overall transfer of coordinate approach to the modeling of wrist-partitioned manipulators see Rill (1985). That work deals (in essence) with the dynamic analysis of a generalized Stewart platform where each leg (manipulator) contains a spherical wrist. Included in the analysis is a simplified (specialized) set of transfer equations which specifically address partitionable manipulators.

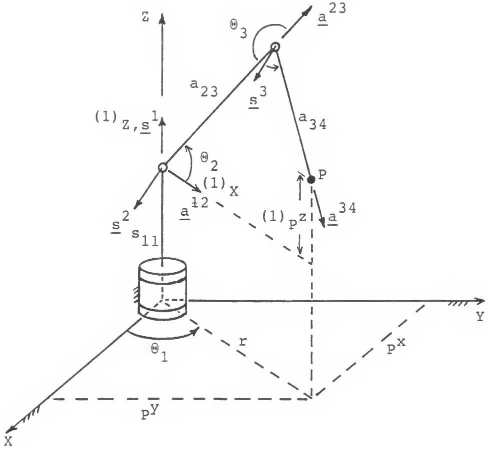


Figure B-1. Wrist-Partitioned Manipulator (First three links)

Table B-1. Kinematic Influence Coefficients for the first three links of the Wrist-Partitioned Manipulator

$\underline{P} = s_{11}\underline{s}^1 + a_{23}\underline{a}^{23} + a_{34}\underline{a}^{34}$

n	$\underline{s}^n$	$\underline{R}^n$	$(\underline{P}-\underline{R}^n)$	g_n^P	$\underline{H}_{in}^P = \underline{H}_{ni}^P$
1	$\underline{s}^1$	$s_{11}\underline{s}^1$	$a_{23}\underline{a}^{23} + a_{34}\underline{a}^{34}$	$\underline{s}^1 \times (\underline{P}-\underline{R}^1)$	$\underline{s}^i \times g_1^P$
2	$\underline{s}^2$	$s_{11}\underline{s}^1$	$a_{23}\underline{a}^{23} + a_{34}\underline{a}^{34}$	$\underline{s}^2 \times (\underline{P}-\underline{R}^2)$	$\underline{s}^i \times g_2^P$
3	$\underline{s}^3$	$s_{11}\underline{s}^1 + a_{23}\underline{a}^{23}$	$a_{34}\underline{a}^{34}$	$\underline{s}^3 \times (\underline{P}-\underline{R}^3)$	$\underline{s}^i \times g_3^P$

$$[G^P] = \begin{bmatrix} -rs\Theta_1 & -(1)p^z c\Theta_1 & -(a_{34}s\Theta_{2+3})c\Theta_1 \\ rc\Theta_1 & -(1)p^z s\Theta_1 & -(a_{34}s\Theta_{2+3})s\Theta_1 \\ 0 & r & a_{34}c\Theta_{2+3} \end{bmatrix}$$

$$[H_{\varphi\varphi}^P]_{1;;} = \begin{bmatrix} -rc\Theta_1 & (1)p^z s\Theta_1 & (a_{34}s\Theta_{2+3})s\Theta_1 \\ (1)p^z s\Theta_1 & -rc\Theta_1 & -(a_{34}c\Theta_{2+3})c\Theta_1 \\ (a_{34}s\Theta_{2+3})s\Theta_1 & -(a_{34}c\Theta_{2+3})c\Theta_1 & -(a_{34}c\Theta_{2+3})c\Theta_1 \end{bmatrix}$$

$$[H_{\varphi\varphi}^P]_{2;;} = \begin{bmatrix} -rs\Theta_1 & -(1)p^z c\Theta_1 & -(a_{34}s\Theta_{2+3})c\Theta_1 \\ -(1)p^z c\Theta_1 & -rs\Theta_1 & -(a_{34}c\Theta_{2+3})s\Theta_1 \\ -(a_{34}s\Theta_{2+3})c\Theta_1 & -(a_{34}c\Theta_{2+3})s\Theta_1 & -(a_{34}c\Theta_{2+3})s\Theta_1 \end{bmatrix}$$

$$[H_{\varphi\varphi}^P]_{3;;} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(1)p^z & -(a_{34}s\Theta_{2+3}) \\ 0 & -(a_{34}s\Theta_{2+3}) & -(a_{34}s\Theta_{2+3}) \end{bmatrix}$$

APPENDIX C

SINGLE-LOOP PLANAR FIVE-BAR ANALYSIS

The analytic example presented here is given as a partial verification of the loop-closure technique developed in the first section of Chapter IV. The well established (Curtis, 1972 and Freeman, 1980) dyad-based linkage analysis procedure is employed for comparison. Consider the five-bar linkage of Fig. C-1.

The results of the dyad-based analysis are given in Table C-1. This analysis is a direct procedure and is summarized as follows

1. Kinematics of dyad poles (i.e., $\underline{A}(t)$ and $\underline{B}(t)$) determined from specified input kinematics ($\varphi_1(t)$ and $\Theta_h(t)$).
2. Kinematics of mass parameters ($X^2(t)$, $Y^2(t)$, ${}^2(t)$) determined using pole kinematics.
3. Pin-joint reaction forces ($\underline{F}^A$, $\underline{F}^B$, $\underline{F}^C$) due to d'Alembert loads determined.
4. Required driving torques (T_1) and (T_h) and reaction forces ($\underline{F}^A$) and (F^B).

Now, following the transfer procedure presented in Chapter IV. one first obtains the open-chain model coefficients

$$S_\varphi = [G^e_\varphi],[H^e_{\varphi\varphi}],[I^*_{\varphi\varphi}],[P^*_{\varphi\varphi\varphi}] \qquad \text{(C-1)}$$

in the manner prescribed in Chapter II, where

$$\underline{e} = (X^h, Y^h, \Theta^h)^T \qquad \text{(C-2)}$$

and

$$\underline{\varphi} = (\varphi_1,\varphi_2,\varphi_3,\varphi_4)^T \tag{C-3}$$

and the resultant coefficients are given in Table C-2. Next, the augmented intermediate coordinate set $\underline{w}$ is chosen to be

$$\underline{w} = \begin{pmatrix} \varphi_1 \\ -\,- \\ \underline{e} \end{pmatrix}, \quad \underline{\alpha} = \varphi_1 \tag{C-4}$$

yielding the corresopnding influence coefficients ($[G^w_\varphi]$) and ($[H^w_{\varphi\varphi}]$) given in Table C-2. Finally, using the transfer equations one obtains

$$\underline{T}_w = \begin{pmatrix} T_1 \\ F^{hx} \\ F^{hy} \\ T_h \end{pmatrix} = [I^*_{ww}]\ddot{\underline{w}} + \dot{\underline{w}}^T[P^*_{www}]\dot{\underline{w}} \tag{C-5}$$

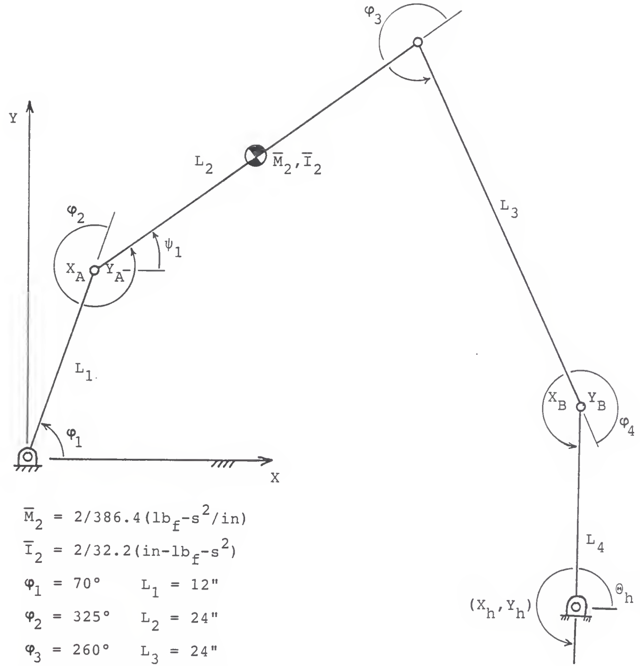
as the generalized load state at the coordinates ($\underline{w}$), where

$$[I^*_{ww}] = [G^w_\varphi]^{-T}[I^*_{\varphi\varphi}][G^w_\varphi]^{-1} \tag{C-6}$$

and

$$[P^*_{www}] = [G^w_\varphi]^{-T}\{([G^w_\varphi]^{-T} \bullet [P^*_{\varphi\varphi\varphi}]) - ([I^*_{ww}] \bullet [H^w_{\varphi\varphi}])\}[G^w_\varphi]^{-1} \tag{C-7}$$

The results of equations (C-5), (C-6) and (C-7) are given in Table C-3. Comparison of these results with those of the direct dyad-based analysis confirm the validity of the transfer approach.



$\overline{M}_2 = 2/386.4(\mathrm{lb_f\text{-}s^2/in})$

$\overline{I}_2 = 2/32.2(\mathrm{in\text{-}lb_f\text{-}s^2})$

$\varphi_1 = 70°$ $L_1 = 12"$

$\varphi_2 = 325°$ $L_2 = 24"$

$\varphi_3 = 260°$ $L_3 = 24"$

$\varphi_4 = 335°$ $L_4 = 12"$

Figure C-1. Planar Five-Bar Mechanism

Table C-1. Dyad-based Analysis

u	Position u (in., rad.)	Velocity u (in/s, rad/s)	Acceleration (u(in/s^2, rad/s^2)
q_1	$\frac{7\pi}{18}$	2.0	1.0
x^A	4.10	-22.55	-27.69
y^A	11.28	8.21	-41.0
q_2	$\frac{1\pi}{2}$	3.0	0.5
x^B	33.9	-36.0	-6.0
y^B	-3.29	0.0	-108.0
1	$\frac{7\pi}{36}$	-0.074	-3.405
x^2	13.93	--	-4.31
y^2	18.16	--	-74.51

$(F^{AX}, F^{AY}) = (-0.009, -0.24) lb_f$ $\quad T_1 = 0.05$ in-lb_f

$(F^{BX}, F^{BY}) = (0.07, -0.15) lb_f$

$(F^{HX}, F^{HY}) = (0.07, -0.15) lb_f$

$T_2 = 0.83$ in-lb_f

Table C-2. Open-chain Model Parameters

$$[G^w_\varphi] = \frac{1\ 0\ 0\ 0}{[G^e_\varphi]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 8.71'' & 19.98'' & 33.75'' & 12'' \\ 33.91'' & 29.80'' & 10.14'' & 0'' \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[H^w_{\varphi\varphi}]_{1;;} = [0] \qquad [H^w_{\varphi\varphi}]_{4;;} = [H^e_{\varphi\varphi}]_{3;;} = [0]$$

$$[H^w_{\varphi\varphi}]_{2;;} = [H^e_{\varphi\varphi}]_{1;;} = -\begin{bmatrix} 33.91 & 29.80 & 10.14 & 0 \\ 29.80 & 29.80 & 10.14 & 0 \\ 10.14 & 10.14 & 10.14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(in.)}$$

$$[H^w_{\varphi\varphi}]_{3;;} = [H^e_{\varphi\varphi}]_{2;;} = \begin{bmatrix} 8.71 & 19.98 & 33.75 & 12.0 \\ 19.98 & 19.98 & 33.75 & 12.0 \\ 33.75 & 33.75 & 33.75 & 12.0 \\ 12.0 & 12.0 & 12.0 & 12.0 \end{bmatrix} \text{(in.)}$$

$$[I^*_{\varphi\varphi}] = \begin{bmatrix} 2.774 & 1.418 & 0 & 0 \\ 1.418 & 0.808 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{in-lb}_f\text{-s}^2)$$

$$[P^*_{\varphi\varphi\varphi}]_{1;;} = \begin{bmatrix} 0 & 0.428 & 0 & 0 \\ 0.428 & 0.428 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{in-lb}_f\text{-s}^2)$$

$$[P^*_{\varphi\varphi}]_{2;;} = \begin{bmatrix} -0.428 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{in-lb}_f\text{-s}^2)$$

$$[P^*\ ]_{3;;} = [0]$$

$$[P^*\ ]_{4;;} = [0]$$

Table C-3. Constrained System Model

$$\underline{T}_W = [I^*_{WW}] \begin{pmatrix} \ddot{q}_1 \\ 0 \\ 0 \\ \ddot{q}_2 \end{pmatrix} + (\dot{q}_1 \ 0 \ 0 \ \dot{q}_2)\ [P^*_{WWW}] \begin{pmatrix} \dot{q}_1 \\ 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 50.77 \times 10^{-3} \text{in-lb}_f \\ 68.83 \times 10^{-3} \text{lb}_f \\ -147.60 \times 10^{-3} \text{lb}_f \\ -825.92 \times 10^{-3} \text{in-lb}_f \end{pmatrix}$$

$$[I^*_{WW}] = \begin{bmatrix} 411.24 & -5.73 & 12.29 & 68.79 \\ -5.73 & 0.26 & -0.55 & -3.10 \\ 12.29 & -0.55 & 1.19 & 6.64 \\ 68.79 & -3.10 & 6.64 & 37.17 \end{bmatrix} \times 10^{-3}$$

$$[P^*_{WWW}]_{1;;} = \begin{bmatrix} 303.35 & -0.18 & -8.96 & 2.15 \\ -0.18 & -0.24 & -0.61 & 2.84 \\ -8.96 & -0.61 & 0.52 & 7.27 \\ 2.15 & 2.84 & 7.27 & -181.56 \end{bmatrix} \times 10^{-3}$$

$$[P^*_{WWW}]_{2;;} = \begin{bmatrix} 3.38 & 0.13 & 0.14 & -1.58 \\ 0.13 & 0.02 & 0.01 & -0.20 \\ 0.14 & 0.01 & 0.005 & 0.17 \\ -1.58 & -0.20 & -0.17 & 9.06 \end{bmatrix} \times 10^{-3}$$

$$[P^*_{WWW}]_{3;;} = \begin{bmatrix} -7.24 & -0.28 & -0.29 & 3.38 \\ -0.28 & -0.04 & -0.03 & 0.43 \\ -0.29 & -0.03 & -0.01 & 0.36 \\ 3.38 & 0.43 & 0.36 & -19.43 \end{bmatrix} \times 10^{-3}$$

$$[P^*_{WWW}]_{4;;} = \begin{bmatrix} -40.50 & -1.58 & -1.66 & 18.95 \\ -1.58 & -0.20 & -0.17 & 2.42 \\ -1.66 & -0.17 & -0.06 & 2.03 \\ 18.95 & 2.42 & 2.03 & -108.75 \end{bmatrix} \times 10^{-3}$$

APPENDIX D

SPECIAL CASE SINGLE-LOOP MECHANISMS

Consider the spatial slider-crank mechanism of Fig. D-1. This device is considered a special case in that it is a two-degree of freedom mechanism, where one of the generalized coordinates (the spin of the coupler link about its axis, Θ_5) has no effect on the output (P^z) motion (Sandor, 1984, Chap. 6). To analyze this mechanism (and others like it) using the unified approach of the first section of Chapter IV, the unspecified (non-actuated) coordinate Θ_5 must be included in all generalized coordinate sets. With this requirement noted, the analysis simply follows the procedure outlined in Fig. D-1 and set forth in Chapter IV.

Consider the Bricard mechanism of Fig. D-2. According to the Gruebler criteria, this mechanism should have zero mobility. However, due to its special geometry, this device has one-degree of freedom (Uicker et al., 1964). The existence of a redundant constraint produces this unexpected freedom and manifests itself in the linear dependence of the screw associated with (or motion afforded by) joint 6 on joints 1 through 5. This means that the Jacobian ($[G_{\varphi}^{u}]$) relating the motion of the hypothetical end-effector ($\underline{u}$) to the initial open-chain generalized coordinates ($\underline{\varphi}$) is of rank 5 and, thus, singular. In terms of the algebra, this

singularity is essential in that it implies the existence of at least one linearly independent non-trivial solution to the equation (see Fig. D-2 for notation)

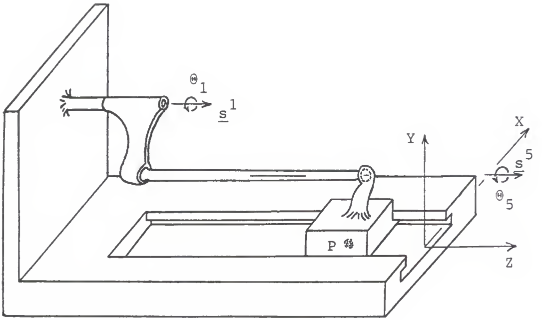
$$\underline{e} = \underline{0} = [{}_1G^{e}_{\varphi}]\,{}_1\underline{\varphi} \qquad \text{(D-1)}$$

Now, from the algebra, since the Jacobian is of dimension 6 and rank 5, there exists exactly one linearly independent solution associated with the one generalized coordinate required to specify the motion of this mechanism. Therefore, equation (D-1) can be solved yielding

$$[{}^{2}G^{\varphi}_{y}] = -([G^{e}_{\varphi}]_{1,2,3,4,5;1,2,3,4,5})^{-1}[G^{e}_{\varphi}]_{1,2,3,4,5;6} = 5\text{x}1 \qquad \text{(D-2)}$$

for the first-order influence coefficients relating the initial coordinates (${}_2\underline{\varphi}$) to the intermediate generalized coordinate ($\underline{y}$). Having obtained these constraining relationships (equation (D-2)) one can obtain the desired dynamic model by, again, following the procedure of Chapter IV. The reader is referred to Fig. D-2 for the basic outline of this analysis. It should be noted that the analysis of this mechanism is problematically similar to having a singular manipulator configuration, yet still being able to attain the specified end-effector motion (i.e., the specified motion space is spanned by the linearly independent columns of the Jacobian). However, here the column dependency is predetermined and the specified motion (rotation by φ_6 about $\underline{s}^6$) is known to be allowable.

Unfortunately, this is not the case in general manipulator motion.



1. $\underline{\varphi} = (\Theta_1, \Theta_2, \ldots, \Theta_7)^T \rightarrow S_\varphi$

$\downarrow$ Transfer

2. $\underline{y} = (\Theta_5, X_p, Y_p, Z_p, \Theta_x, \Theta_y, \Theta_z)^T \rightarrow S_y$

$\downarrow$ Constrain

3. $\underline{w} = (\Theta_5, Z_p)^T \rightarrow S_w$

$\downarrow$ Transfer

4. $q = (\Theta_1, \Theta_5)^T \rightarrow S_q$

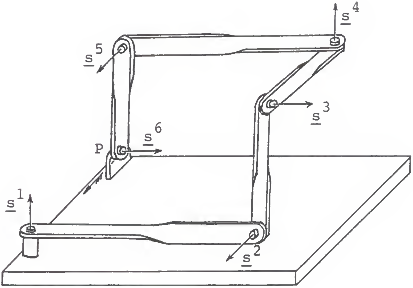
Figure D-1. Spatial slider-crank mechanism

$$_1\underline{\varphi} = (\Theta_1, \Theta_2, \ldots, \Theta_6)^T$$

$$\underline{\dot{e}} = \begin{pmatrix} \underline{\dot{P}} \\ \underline{\omega}^{67} \end{pmatrix}$$

$$_2\underline{\varphi} = (\Theta_1, \Theta_2, \ldots, \Theta_5)^T$$

$$\underline{y} = \Theta_6$$



1. Determine $[_1G^e_{\varphi}], [_1H^e_{\varphi\varphi}]$

2. Determine $_2S_{\varphi} = [_2I^*_{\varphi\varphi}], [_2P^*_{\varphi\varphi\varphi}], [_2H^y_{\varphi\varphi}]$

3. Determine $[^2G^{\varphi}_y]$ from equation (D-2)
   and $[^2H^{\varphi}_{yy}]$ from equation (3-19)

4. Obtain intermediate model
   $$S_y = [I^*_{yy}], [P^*_{yyy}], \underline{T}_y$$

Figure D-2. The Bricard mechanism

APPENDIX E

PARALLEL MECHANISM MODELING

Consider the three-degree of freedom parallel-input planar robotic mechanism of Fig. E-1. The goal of this analysis is to verify the overall modeling procedure presented in the second section of Chapter IV, particularly the distribution of the system mass parameters. To accomplish this, the system is specified such that the desired model coefficients are directly available, the transfer procedure is then applied and the results obtained are compared with the known (directly obtrained) coefficient values.

The desired input locations ($\underline{q}$) are specified to be the base joints of the three branches

$$\underline{q} = ({}_1\varphi_1,\ {}_2\varphi_1,\ {}_3\varphi_1)^T \tag{E-1}$$

Now, in order to have direct knowledge of the desired model only the base links themselves can have mass. Refering to the specifications given in Fig. E-1, the effective inertia matrix referenced to the desired coordinates ($\underline{q}$) is

$$[I^*_{qq}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{constant} \tag{E-2}$$

and the velocity related coefficient ($[P^*_{qqq}]$)is identically zero (i.e., each actuator is solely responsible for its own

mass). With the final (desired) model known, the transfer technique is now applied to verify its accuracy. First, following the analyses of Chapter II, the (independently treated) joint referenced model for each leg is obtained

$$({}_rS_\varphi) = [{}_rG^w_\varphi],[{}_rH^w_{\varphi\varphi}],\ [{}_rI^*_{\varphi\varphi}]\ ,\ r = 1,2,3 \tag{E-3}$$

The results of equation (E-3) for branch one (i.e., r = 1) are given in Table E-1. Then, applying the transfer equations (4-43), (4-44), (4-45) and (4-46), the system model referenced to the intermediate generalized coordinates ($\underline{w}$) is determined, yielding (see Table E-1)

$$(S_w) = [{}^rG^\varphi_w],[{}^rH^\varphi_{ww}],[{}^rI^*_{ww}],[{}^rP^*_{www}] \tag{E-4}$$

Next, the effects of each leg are combined and the influence coefficients relating desired coordinates ($\underline{q}$) to the intermediate coordinates ($\underline{w}$) are formed, giving (see Table E-2)

$$[I^*_{ww}] = \sum_{r=1}^{3}[{}^rI^*_{ww}]$$

$$[P^*_{www}] = \sum_{r=1}^{3}[{}^rP^*_{www}] \tag{E-5}$$

and

$$[G^q_w] = \begin{bmatrix} [{}^1G^\varphi_w]_{1;} \\ [{}^2G^\varphi_w]_{1;} \\ [{}^3G^\varphi_w]_{1;} \end{bmatrix} \tag{E-6}$$

$$[H^q_{ww}] = \begin{bmatrix} [{}^1H^{\varphi}_{ww}]_1; \\ [{}^2H^{\varphi}_{ww}]_1; \\ [{}^3H^{\varphi}_{ww}]_1; \end{bmatrix}$$

Finally, applying the transfer equations once more, the expected desired model

$$[I^*_{qq}] = [G^q_w]^{-T}[I^*_{ww}][G^q_w]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{E-7}$$

$$[P^*_{qqq}] = [G^q_w]^{-T}\{([G^q_w]^{-T} \bullet [P^*_{www}]) - ([I^*_{ww}] \bullet [H^q_{ww}])\}[G^q_w]^{-1}$$

$$= [0]$$

is obtained, verifying the transfer technique.

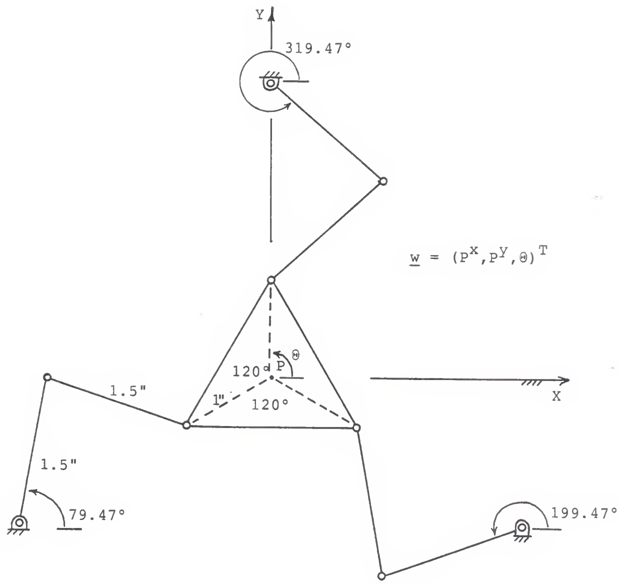


Figure E-1. Three-DOF Planar Robotic Mechanism

Table E-1. Joint- and Hand-Referenced Models (r=1)

$$[{}_1G^{w}_{\varphi}] = \begin{bmatrix} -1.475 & 0 & -0.5 \\ 2.55 & 2.28 & 0.866 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \text{(in.)} \\ \text{(in.)} \\ \; \end{matrix} \qquad [{}_1I^{*}_{\varphi\varphi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(in-lb}_f\text{s}^2\text{)}$$

$$[{}_1H^{w}_{\varphi\varphi}]_{1;;} = \begin{bmatrix} -2.55 & -2.28 & -0.866 \\ -2.28 & -2.28 & -0.866 \\ -0.866 & -0.866 & -0.866 \end{bmatrix} \text{(in)} \qquad [{}_1H^{w}_{\varphi\varphi}]_{3;;} = [0]$$

$$[{}_1H^{w}_{\varphi\varphi}]_{2;;} = \begin{bmatrix} -1.475 & 0 & -0.5 \\ 0 & 0 & -0.5 \\ -0.5 & -0.5 & -0.5 \end{bmatrix} \text{(in.)} \qquad [{}_1P^{*}_{\varphi\varphi\varphi}] = [0]$$

$$[{}_1G^{\varphi}_{w}] = \begin{bmatrix} -0.637 & 0.225 & -0.513 \\ 0.758 & 0.439 & -0.001 \\ -0.122 & -0.664 & 1.514 \end{bmatrix} \begin{matrix} \text{(in.}^{-1}\text{)} \\ \text{(in.}^{-1}\text{)} \\ \; \end{matrix} \qquad [{}^1I^{*}_{ww}] = \begin{bmatrix} 0.405 & -0.143 & 0.327 \\ -0.143 & 0.051 & -0.116 \\ 0.327 & -0.116 & 0.264 \end{bmatrix}$$

$$[{}^1H^{\varphi}_{ww}]_{1;;} = \begin{bmatrix} 0.050 & -0.105 & 0.116 \\ -0.105 & -0.439 & 0.328 \\ 0.116 & 0.328 & -0.665 \end{bmatrix} \text{(in.}^{-1}\text{)}$$

$$[{}_1P^{*}_{www}]_{1;;} = \begin{bmatrix} -0.032 & 0.067 & -0.074 \\ 0.067 & 0.280 & -0.209 \\ -0.074 & -0.209 & 0.423 \end{bmatrix}$$

$$[{}_1H^{\varphi}_{ww}]_{2;;} = \begin{bmatrix} 0.358 & -0.053 & 0.225 \\ -0.053 & 0.419 & -0.390 \\ 0.225 & -0.390 & 1.327 \end{bmatrix} \text{(in.}^{-1}\text{)}$$

$$[{}^1P^{*}_{www}]_{2;;} = \begin{bmatrix} 0.011 & -0.024 & -0.026 \\ -0.024 & -0.099 & 0.074 \\ 0.026 & 0.074 & -0.150 \end{bmatrix}$$

$$[{}^1H^{\varphi}_{ww}]_{3;;} = \begin{bmatrix} -0.408 & 0.158 & -0.341 \\ 0.158 & 0.019 & 0.062 \\ -0.341 & 0.062 & -0.662 \end{bmatrix}$$

$$[{}^1P^{*}_{www}]_{3;;} = \begin{bmatrix} -0.026 & 0.054 & -0.059 \\ 0.054 & 0.225 & -0.168 \\ -0.059 & -0.168 & 0.341 \end{bmatrix}$$

Table E-2. Complete Hand-Referenced Model

$$[G^q_w] = \begin{bmatrix} -0.637 & 0.225 & -0.513 \\ 0.124 & -0.665 & -0.514 \\ 0.513 & 0.439 & -0.513 \end{bmatrix} \begin{matrix} (\text{in.}^{-1}) \\ (\text{in.}^{-1}) \\ \end{matrix} \quad [I^*_{ww}] = \begin{bmatrix} 1.225 & 0.367 & -0.589 \\ 0.367 & 1.513 & -0.106 \\ -0.589 & -0.106 & 1.582 \end{bmatrix}$$

$$[H^q_{ww}]_{1;;} = \begin{bmatrix} 0.050 & -0.105 & 0.116 \\ -0.159 & -0.439 & 0.326 \\ 0.116 & 0.326 & -0.665 \end{bmatrix} (\text{in.}^{-1})$$

$$[P^*_{www}]_{1;;} = \begin{bmatrix} -0.479 & 0.432 & 0.188 \\ 0.432 & 0.031 & -0.629 \\ 0.188 & -0.629 & -0.762 \end{bmatrix}$$

$$[H^q_{ww}]_{2;;} = \begin{bmatrix} -0.408 & -0.159 & -0.342 \\ -0.159 & 0.018 & -0.633 \\ -0.342 & -0.633 & -0.665 \end{bmatrix} (\text{in.}^{-1})$$

$$[P^*_{www}]_{2;;} = \begin{bmatrix} 0.259 & 0.534 & 0.777 \\ 0.534 & -0.340 & -0.188 \\ 0.777 & -0.188 & -0.137 \end{bmatrix}$$

$$[H^q_{ww}]_{3;;} = \begin{bmatrix} -0.225 & 0.263 & 0.225 \\ 0.263 & -0.165 & -0.263 \\ 0.225 & -0.263 & -0.664 \end{bmatrix}$$

$$[P^*_{www}]_{3;;} = \begin{bmatrix} 0.741 & -0.188 & -0.054 \\ -0.188 & 0.459 & 0.302 \\ -0.054 & 0.302 & 2.045 \end{bmatrix}$$

Units for $[I^*_{ww}]$ and $[P^*_{www}]$ include: $(\text{in-lb}_f\text{-s}^2)$ - I^*_{33}

$(\text{lb}_f\text{-s}^2)$ - I^*_{13}

$(\text{lb}_f\text{-S}^2/\text{in.})$ - I^*_{11}

BIBLIOGRAPHY

Armada, M.A., and No, J. 1981. Bilinear manipulator models: Application to suboptimal control. Proc. 11th Int. Symp. Industrial Robots 747-754.

Assada, H., and Youcef-Toumi, K. 1983. Analysis and design of semi-direct-drive robot arms. Proc. of 1983 American Contr. Conf. 757-764.

Behi, F. 1985. Ph.D. dissertation in preparation, University of Florida.

Benedict, C.E., and Tesar, D. 1971. Dynamic response analysis of quasi-rigid mechanical systems using kinematic influence coefficients. J. Mechanisms 6:383-403.

Benedict, C.E., and Tesar, D. 1978a(Oct.). Model formulation of complex mechanisms with multiple inputs: Part I - Geometry. J. Mechanical Design 100(4):747-753.

Benedict, C.E., and Tesar, D. 1978b(Oct.). Model formulation of complex mechanisms with multiple inputs: Part II - The dynamic model. J. Mechanical Design 100(4): 755-761.

Borovac, B., Vukobratovic, M., and Stokic, D. 1983. Analysis of the influence of actuator model complexity on manipulator control synthesis. Mechanism Mach. Theory 18(2):113-122.

Cox, D.J., and Tesar, D. 1981(Sept.). The dynamic modeling and command signal formulation for parallel multi-parameter robotic devices. Internal report, CIMAR, University of Florida.

Curtis, F.C. 1972. Dynamic state analysis of mechanisms composed of binary structury groups. Master's thesis, University of Florida.

Dubowsky, S., and DesForges, D.T. 1979(Sept.). The application of model-referenced adaptive control to robotic manipulators. Trans. ASME J. Dyn. Syst., Measurement Contr. 101(3):193-200.

Duffy, J. 1980. Analysis of mechanisms and robot manipulators. New York, N.Y.: John Wiley and Sons, Inc.

Duffy, J., and Crane, C. 1980. A displacement analysis of the general spatial 7-link, 7R mechanism. Mechanism Mach. Theory 15(3):153-169.

Electrocraft. 1978. DC motors, speed controls, servo systems. Hopkins, Minn.: Electro-Craft Corp.

Freeman, R.A. 1980. Interchange of reference parameters for dynamic modeling and analysis of complex multi-degree of freedom planar mechanical systems. Master's thesis, University of Florida.

Freeman, R.A., Tesar, D. 1982(Jan.). The generalized coordinate selection for the dynamics of complex planar mechanical systems. Trans. ASME, J. Mechanical Design 104(1):206-217.

Freeman, R.A., Tesar, D. 1983. Dynamic model for generalized planar robotic manipulator. Proc. 6th World Congress on Theory of Mach. and Mechanisms 2:998-1002.

Fresonke, D. 1985 (May). Analytic formulation for deformation compensation in robotic structures. Master's thesis, University of Florida.

Freund, E. 1982. Fast nonlinear control with arbitrary pole-placement for industrial robots and manipulators. Int. J. Robotics Research 1(1):65-78.

Golla, D.F., Garg, S.C., and Hughes, P.C. 1981. Linear state-feedback control of manipulators. Mech. Machine Theory 16(1):93-103.

Guez, A. 1982. Optimal control of N degrees of freedom robotic manipulator. Ph.D. dissertation, University of Florida.

Hewit, J.R., and Burdess, J.S. 1981. Fast dynamic decoupled control for robotics, using active force control Mech. Machine Theory 16(5):535-542.

Hollerbach, J.M. 1980. A recursive formulation of Lagrangian manipulator dynamics. IEEE Trans. Syst. Man Cybern. SMC - 10(11):730-736.

Hollerbach, J.M., and Sahar, G. 1983. Wrist-partitioned, inverse kinematic accelerations and manipulator dynamics. Int. J. Robotics Res. 2(4):61-76.

Horowitz, R., and Tomizuka, M. 1980(Nov. 16-21). An adaptive control scheme for mechanical manipulators-compensation of nonlinearity and decoupling control. Presented Winter Annual Meeting Dyn. Syst. Contr. division of ASME.

Kahn, M.E., and Roth, B. 1971. The near minimum-time control of open-loop articulated kinematic chains. Trans. ASME J. Dyn. Syst., Measurement Contr. 93:164-172.

Kane, T.R. 1961. Dynamics of nonholonomic systems. J. Applied Mechanics 28:574-578.

Kane, T.R., and Levinson, D.A. 1983. The use of Kane's dynamic equations in robotics. Int. J. Robotics Res. 2(3):3-21.

Khatib, O. 1983. Dynamic control of manipulators in operational spaces. Proc. 6th World Congress on Theory of Mach. and Mechanisms 2:1128-1131.

Lanczos, C. 1962. The variational principles of mechanics. Toronto: University of Toronto Press.

LeBorgne, M., Ibarra, J.M., and Espiau, B. 1981. Adaptive control of high velocity manipulators. 11th Int. Symp. Industrial Robots 227-236.

Lee, C.S.G. 1982(Dec.). Robot arm kinematics, dynamics, and control. IEEE Computer 15(12):62-80.

Lee, C.S.G., and Lee, B.H. 1984(June). Resolved motion adaptive control for mechanical manipulators. Trans. ASME J. Dyn. Syst., Measurement Contr. 106:134-142.

Liegeois, A., Fournier, A., and Aldon, M.J. 1980(Aug. 13-15). Model reference control of high-velocity industrial robots. Proc. 1980 Joint Automatic Control Conf. TPIO-D.

Luh, J.Y.S., Walker, M.W., and Paul, R.P.C. 1980(June). Resolved acceleration control of mechanical manipulators. IEEE Trans. Auto. Contr. 25(3):468-474.

Meirovitch, L. 1970. Methods of analytical dynamics. New York: McGraw-Hill Book Company.

Myklebust, A. 1982(Jan.). Dynamic response of an electric motor-linkage system during startup. Trans. ASME J. Mechanical Design 104(1):137-142.

Orin, D.E., and McGhee, R.B. 1981(Sept.). Dynamic computer simulation of robotic mechanisms. Presented Conf. Theory and Practice of Robots and Manipulators, Warsaw, Poland.

Paul, R.P. 1972(Nov.). Modelling, trajectory calculation, and seroing of a computer controlled arm. AIM 177, Stanford University, Artificial Intelligence Laboratory.

Paul, R.P., Shimano, B., and Mayer, G.E. 1981a(June). Kinematic control equations for simple manipulators. IEEE Trans. Syst. Man Cybern. SMC-11(6):449-455.

Paul, R.P., Shimano, B., and Mayer, G.E. 1981b(June). Differential kinematic control equations for simple manipulators. IEEE Trans. Syst. Man Cybern. SMC-11(6):456-460.

Raibert, M.H., and Craig, J.J. 1981(June). Hybrid position/force control of manipulators. Trans. ASME J. Dyn. Syst., Measurement Contr. 102:126-133.

Rill, D. 1985. Master's thesis in preparation, University of Florida.

Sandor, G.N., and Erdman, A.G. 1984. Advanced mechanism design: Analysis and synthesis 2. London: Prentice-Hall International, Inc.

Silver, W. 1982. On the equivalence of Lagrangian and Newton-Euler dynamics for manipulators. Int. J. Robotics Research 1(2):60-70.

Sklar, M., and Tesar, D. 1984(Apr.). The analysis of hybrid parallel/serial manipulators. Internal report, CIMAR, University of Florida.

Stoten, D.P. 1983. The adaptive control of manipulator arms. Mechanism Mach. Theory 18(4):283-288.

Sunada, W.H., and Dubowsky, S. 1983. On the dynamic analysis and behavior of industrial robotic manipulators with elastic members. To appear Trans. ASME J. Mechanical Design. Paper No. 82-DET-45.

Tesar, D., and Thomas, M. 1979. Assessment for the mathematical formulation for the design and digital control of programmable manipulator systems. NSF Grant ENG78-20012 and DOE Contract Er-78-5-05-6102. CIMAR, University of Florida.

Tesar, D., and Kamen, E. 1983(Jan.). Control-in-the-small technology for enhanced precision under load for light machining with industrial robots. Preliminary proposal, CIMAR, University of Florida.

Thomas, M. 1981. Dynamic model formulation for rigid-link serial manipulators with revolute joints. Master's thesis, University of Florida.

Thomas, M., and Tesar, D. 1982a(Nov.). Dynamic modeling and analysis of rigid-link serial manipulators. DOE Grant De-AC05-79ER10013. CIMAR, University of Florida.

Thomas, M., and Tesar, D. 1982b(Sept.). Dynamic modeling of serial manipulator arms. Trans. ASME J. Dyn. Syst., Measurement Contr. 104(3):218-228.

Thomas, M., Yuan-Chou, H.C., and Tesar, D. 1984(Oct.). Optimal actuator sizing for robotic manipulators based on local dynamic criteria. Submitted to ASME, Mechanisms Conf., Cambridge, MA.

Thomas, M., Yuan-Chou, H.C., and Tesar, D. 1985(Mar. 25-28). Trans. IEEE Int. Conf. Robotics Automation 275-281.

Uicker, J.J., Denavit, J., and Hartenburg, R.S. 1964(June). An iterative method for the displacement analysis of spatial mechanisms. Trans. ASME Series E 86:309-314.

Vukobratovic, M., and Stokic, D. 1982a. Control of manipulation robots: Theory and application. (Monograph.) New York: Springer-Verlag.

Vukobratovic, M., and Kircanski, N. 1982b. Computer-oriented method for linearization of dynamic models of active spatial mechanisms. Mechanism Mach. Theory 17(1):21-32.

Walker, M.W., and Orin, D.E. 1982. Efficient dynamic computer simulation of robot mechanisms. Trans. ASME J. Dyn. Syst., Measurement Contr. 104(3):205-211.

Wander, J.P. 1985. Master's thesis in preparation, University of Florida.

Whitehead, M.L. 1984. Control of serial manipulators with emphasis on disturbance rejection. Master's thesis, University of Florida.

Whitney, D.E. 1969(June). Resolved motion rate control of manipulators and human prostheses. IEEE Trans. Man Mach. Syst. MMS-10(2):47-53.

Whitney, D.E. 1972(Dec.). The mathematics of coordinated control of prosthetic arms and manipulators. Trans. ASME J. Dyn. Syst., Measurement Contr. 94(4):303-309.

Wittenbauer, F. 1923. Graphische Dynamik. Berlin: Springer.

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Robert Arthur Freeman was born in Gainesville, Florida, on August 26, 1954. He grew up in Gainesville, Florida, and attended P.K. Yonge Laboratory School in Gainesville, graduating in June of 1972. He entered the University of Florida in September of 1972, receiving a Bachelor of Science in Mechanical Engineering in August of 1976 and a Master of Engineering in March of 1980. He was employed as a design engineer for the Wayne H. Coloney Company in Tallahassee, Florida, from September of 1979 to September of 1980. He reentered the University of Florida in September of 1980 and began working towards the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Delbert Tesar, Chairman
Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentaiton and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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